FIRST-ORDER DIFFERENTIAL CALCULI OVER MULTI-BRAIDED QUANTUM GROUPS

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ABSTRACT. A differential calculus of the first order over multi-braided quantum groups is developed. In analogy with the standard theory, left/right-covariant and bicovariant differential structures are introduced and investigated. Furthermore, antipodally covariant calculi are studied. The concept of the *-structure on a multi-braided quantum group is formulated, and in particular the structure of left-covariant *-covariant calculi is analyzed. A special attention is given to differential calculi covariant with respect to the action of the associated braid system. In particular it is shown that the left/right braided-covariance appears as a consequence of the left/right-covariance relative to the group action. Braided counterparts of all basic results of the standard theory are found.

CONTENTS

1. Introduction

The basic theme of this study is the analysis of the first-order differential structures over multi-braided quantum groups. Standard braided quantum groups [4] are included as a special case into the theory of multi-braided quantum groups [3]. The difference between two types of braided quantum groups is in the behavior of the coproduct map. In the standard theory, the coproduct $\phi\colon \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ is interpretable as a morphism in a braided category generated by the basic algebra $\mathcal{A}$ and the associated braiding $\sigma\colon \mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$. In our generalized framework two standard pentagonal diagrams expressing compatibility between ϕ and σ are replaced by a single more general octagonal diagram.

It then turns out that the lack of the functoriality of the coproduct map is 'measurable' by a second braid operator $\tau\colon \mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$. Furthermore, two braid operators generate in a natural manner a generally infinite system of braid operators $\sigma_n\colon \mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$, where $n \in \mathbb{Z}$, which elegantly express twisting properties of all the maps appearing in the game. This explains our attribute *multi*-braided, for the structures we are dealing with.

Multi-braided quantum groups include various completely 'pointless' structures, overcoming in such a way an inherent geometrical inhomogeneity of standard quantum groups and braided quantum groups. This inhomogeneity is explicitly visible in geometrical situations in which 'diffeomorphisms' of quantum spaces appear. For example, in the theory of locally trivial quantum principal bundles over classical smooth manifolds [2] a natural correspondence between quantum G-bundles (where G is a standard compact quantum group) and ordinary G_{cl}-bundles (over the same manifold) holds. Here G_{cl} is the classical part of G.

The multi-braided formalism reduces to the standard braided quantum groups iff $\sigma = \tau$, which means that all the operators σ_n coincide with σ. This is also equivalent to the multiplicativity of the counit map.

In the formalization of the concept of a first-order calculus, we shall follow [5]: If the algebra $\mathcal{A}$ represents a quantum space X, then every first-order calculus over X will be represented by an $\mathcal{A}$-bimodule Γ, playing the role of the 1-forms on X, together with a standard derivation $d : \mathcal{A} \to \Gamma$, playing the role of the differential. Such a formalization reflects noncommutative-geometric [1] philosophy, according to which the concept of a differential form should be the starting point for a foundation of the quantum differential calculus.

The paper is organized as follows. In Section 2 we first study differential calculi over a quantum space X, compatible in the appropriate sense with a single braid operator $\sigma\colon \mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$. In this context, left/right, and bi-σ-covariant differential structures are distinguished. The notion of left σ-covariance requires a natural extendability of σ to a flip-over operator $\sigma^{\mathrm{l}}\colon \Gamma \otimes \mathcal{A} \to \mathcal{A} \otimes \Gamma$. Similarly, right σ-covariance requires extendability of σ to a flip-over operator $\sigma^{\mathrm{r}}\colon \mathcal{A} \otimes \Gamma \to \Gamma \otimes \mathcal{A}$. Finally, the concept of bi-σ-covariance is simply a symbiosis of the previous two.

We shall then briefly analyze general situations in which the calculus is covariant relative to a given braid system $\mathcal{T}$ operating in $\mathcal{A}$.

At the end of Section 2 we begin the study of differential structures over multi-braided quantum groups. We shall prove that if $\mathcal{A}$ and σ are associated to a multi-braided quantum group G then the left/right σ-covariance implies the left/right σ_n-covariance, for each $n \in \mathbb{Z}$. We shall also analyze interrelations between all

possible flip-over operators and maps determining the group structure.

All considerations with braid operators can be performed at the language of braid and tangle diagrams, as in the framework of braided categories [4]. The unique additional moment is that crossings of diagrams should be appropriately labeled, since we are in a multi-braided situation. At the diagramatic level, many of the proofs become very simple. However, in this study the considerations will be performed in the standard-algebraic way. For the reasons of completeness, all the proofs are included in the paper.

Through sections 3–5 we shall exclusively deal with a given multi-braided quantum group G. In Section 3 we begin with formulations of braided counterparts of concepts of the left, right and bi-covariance [5]. As in the standard theory [5], the notion of left covariance will be formulated by requiring a possibility of defining a left action $\ell_\Gamma \colon \Gamma \to \mathcal{A} \otimes \Gamma$ of G on Γ. Similarly, right covariance will be characterized by a possibility of a right action $\varrho_\Gamma \colon \Gamma \to \Gamma \otimes \mathcal{A}$. The notion of bicovariance is a symbiosis of the previous two.

Our attention will be then confined to the left-covariant structures. As we shall see, left covariance implies left σ-covariance and, consequently, left σ_n-covariance, for each $n \in \mathbb{Z}$. The corresponding flip over operators σ_n^{l} naturally describe twisting properties of the left action ℓ_Γ. Besides the study of properties of maps ℓ_Γ and σ_n^{l}, and their interrelations, we shall also analyze the internal structure of left-covariant calculi. It turns out that the situation is more or less the same as in the standard theory [5]. As a left/right $\mathcal{A}$-module, every left-covariant Γ is free and can be invariantly decomposed as

$$\Gamma \leftrightarrow \mathcal{A} \otimes \Gamma_{inv} \leftrightarrow \Gamma_{inv} \otimes \mathcal{A}$$

where Γ_{inv} is the space of left-invariant elements of Γ. We shall also prove a braided generalization of the structure theorem [5] by establishing a natural correspondence between (classes of isomorphic) left-covariant Γ and certain lineals $\mathcal{R} \subseteq \ker(\epsilon)$.

However, a full analogy with [5] breaks, because $\mathcal{R}$ is generally not a right ideal in $\mathcal{A}$, but a right ideal in a simplified [3] algebra $\mathcal{A}_0$ obtained from $\mathcal{A}$ by an appropriate change of the product. The lineal $\mathcal{R}$ should also be left-invariant with respect to the action of τ.

Concerning the concept of the right covariance, it is in some sense symmetric to that of the left covariance. For this reason, we shall not repeat completely analogous considerations for right-covariant calculi. The most important properties of them are collected, without proofs, in Appendix A. In particular, right covariance implies right σ_n-covariance, for each $n \in \mathbb{Z}$.

The study of bicovariant differential structures is the topic of Section 4. In the bicovariant case the action maps ℓ_Γ and ϱ_Γ, as well as the flip-over maps $\sigma_n^{\mathrm{l}}, \sigma_n^{\mathrm{r}}$ are mutually compatible, in a natural manner.

We shall characterize bicovariance in terms of the corresponding right $\mathcal{A}_0$-ideals $\mathcal{R}$. It turns out that the calculus is bicovariant if and only if $\mathcal{R}$ satisfies two additional conditions. The first one correspond to the adjoint invariance in the standard theory [5]. In its formulation, a braided analogue of the adjoint action of G on itself appears naturally. For this reason, the most important properties of this map are collected in Appendix B. The second additional condition for $\mathcal{R}$, trivial in the standard theory, consists in its right τ-invariance.

In Section 5, we shall analyze differential structures which are covariant with respect to the antipodal map $\kappa\colon \mathcal{A} \to \mathcal{A}$. Such structures will be called κ-covariant. As we shall see, a symbiosis of κ and left covariance is equivalent to bicovariance.

In Section 6 we shall first introduce the concept of a *-structure on a multi-braided quantum group. Then, we pass to the study of *-covariant calculi, in the context of multi-braided quantum groups.

Besides other results we shall obtain a characterization of *-covariant left covariant structures Γ, in terms of the corresponding right $\mathcal{A}_0$-ideal $\mathcal{R}$. It turns out that a left-covariant calculus is *-covariant iff $\kappa(\mathcal{R})^* \subseteq \mathcal{R}$, which is identical as in the standard theory [5].

In this paper only the abstract theory will be presented. Concrete examples will be included in the next part of the study, after developing a higher-order differential calculus. This will include differential structures over already considered groups, as well as new examples of 'differential' multi-braided quantum groups coming from the developed theory. Finally, let us mention that we shall assume here *trivial* braiding properties of the differential $d\colon \mathcal{A} \to \Gamma$. Our philosophy is that the non-trivial braidings involving the differential map should be interpreted as an extra structure given over the whole differential calculus.

Acknowledgment

2. The Concept of Braided Covariance

Let $\mathcal{A}$ be a complex unital associative algebra. Let us denote by $m\colon \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ the multiplication in $\mathcal{A}$. The algebra $\mathcal{A}$ will be interpreted as consisting of smooth functions over a quantum space X.

By definition, a first order differential calculus over X is a unital $\mathcal{A}$-bimodule Γ, equipped with a linear map $d\colon \mathcal{A} \to \Gamma$ satisfying the Leibniz rule

$$(2.1) \qquad\qquad dm = m_\Gamma^l(\mathrm{id} \otimes d) + m_\Gamma^r(d \otimes \mathrm{id}),$$

and such that $\iota_\Gamma^l = m_\Gamma^l(\mathrm{id} \otimes d)\colon \mathcal{A} \otimes \mathcal{A} \to \Gamma$ is surjective. Here, $m_\Gamma^l\colon \mathcal{A} \otimes \Gamma \to \Gamma$ and $m_\Gamma^r\colon \Gamma \otimes \mathcal{A} \to \Gamma$ are the left and the right $\mathcal{A}$-module structures of Γ.

Let us observe that $d(1) = 0$, and that the surjectivity of ι_Γ^l is equivalent to the surjectivity of $\iota_\Gamma^r\colon \mathcal{A} \otimes \mathcal{A} \to \Gamma$, which is given by $\iota_\Gamma^r = m_\Gamma^r(d \otimes \mathrm{id})$.

Now, let us assume that X is a *braided* quantum space. In other words, we have in addition a bijective braid operator $\sigma\colon \mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ such that the following identities hold:

$$(2.2) \qquad\qquad (\mathrm{id} \otimes m)(\sigma \otimes \mathrm{id})(\mathrm{id} \otimes \sigma) = \sigma(m \otimes \mathrm{id})$$
$$(2.3) \qquad\qquad (m \otimes \mathrm{id})(\mathrm{id} \otimes \sigma)(\sigma \otimes \mathrm{id}) = \sigma(\mathrm{id} \otimes m).$$

The operator σ naturally induces a structure of an associative algebra on $\mathcal{A} \otimes \mathcal{A}$ with the unit element $1 \otimes 1 \in \mathcal{A} \otimes \mathcal{A}$. Explicitly, the product is given by

$$(a \otimes b)(q \otimes d) = a\sigma(b \otimes q)d.$$

We are going to analyze natural compatibility conditions between Γ and σ.

Definition 1. A first-order differential calculus Γ over X is called left σ-covariant iff there exists a linear operator $\sigma^{\mathrm{l}} \colon \Gamma \otimes \mathcal{A} \to \mathcal{A} \otimes \Gamma$ satisfying

$$(2.4) \qquad \sigma^{\mathrm{l}}(\iota_\Gamma^{\mathrm{l}} \otimes \mathrm{id}) = (\mathrm{id} \otimes \iota_\Gamma^{\mathrm{l}})(\sigma \otimes \mathrm{id})(\mathrm{id} \otimes \sigma).$$

Similarly, we say that Γ is right σ-covariant iff there exists a linear operator $\sigma^{\mathrm{r}} \colon \mathcal{A} \otimes \Gamma \to \Gamma \otimes \mathcal{A}$ such that

$$(2.5) \qquad \sigma^{\mathrm{r}}(\mathrm{id} \otimes \iota_\Gamma^{\mathrm{r}}) = (\iota_\Gamma^{\mathrm{r}} \otimes \mathrm{id})(\mathrm{id} \otimes \sigma)(\sigma \otimes \mathrm{id}).$$

Finally, Γ is called bi-σ-covariant iff it is both right and left σ-covariant.

The idea beyond this definition is that 'twistings' between elements from $\mathcal{A}$ and Γ are performable 'term by term' such that twistings between the symbol d and elements from $\mathcal{A}$ are trivial.

It is easy to see that maps σ^{l} and σ^{r}, if they exist, are uniquely determined by (2.4) and (2.5) respectively.

Requirement (2.4) can be replaced by the equivalent

$$(2.6) \qquad \sigma^{\mathrm{l}}(\iota_\Gamma^{\mathrm{r}} \otimes \mathrm{id}) = (\mathrm{id} \otimes \iota_\Gamma^{\mathrm{r}})(\sigma \otimes \mathrm{id})(\mathrm{id} \otimes \sigma).$$

Similarly, the operator σ^{r} appearing in the context of the right σ-covariance can be characterized by

$$(2.7) \qquad \sigma^{\mathrm{r}}(\mathrm{id} \otimes \iota_\Gamma^{\mathrm{l}}) = (\iota_\Gamma^{\mathrm{l}} \otimes \mathrm{id})(\mathrm{id} \otimes \sigma)(\sigma \otimes \mathrm{id}).$$

In the following proposition, the most important general properties of σ-covariant structures are collected.

Proposition 2.1. *(i) If Γ is a left σ-covariant calculus, then*

$$(2.8) \qquad \sigma^{\mathrm{l}}(\vartheta \otimes 1) = 1 \otimes \vartheta \qquad \sigma^{\mathrm{l}}(d \otimes \mathrm{id}) = (\mathrm{id} \otimes d)\sigma$$

$$(2.9) \qquad (\mathrm{id} \otimes \sigma^{\mathrm{l}})(\sigma^{\mathrm{l}} \otimes \mathrm{id})(\mathrm{id} \otimes \sigma) = (\sigma \otimes \mathrm{id})(\mathrm{id} \otimes \sigma^{\mathrm{l}})(\sigma^{\mathrm{l}} \otimes \mathrm{id}).$$

The map σ^{l} is surjective. Its kernel is an $\mathcal{A}$-subbimodule of $\Gamma \otimes \mathcal{A}$. Furthermore, we have

$$(2.10) \qquad (\mathrm{id} \otimes m_\Gamma^{\mathrm{l}})(\sigma \otimes \mathrm{id})(\mathrm{id} \otimes \sigma^{\mathrm{l}}) = \sigma^{\mathrm{l}}(m_\Gamma^{\mathrm{l}} \otimes \mathrm{id})$$

$$(2.11) \qquad (\mathrm{id} \otimes m_\Gamma^{\mathrm{r}})(\sigma^{\mathrm{l}} \otimes \mathrm{id})(\mathrm{id} \otimes \sigma) = \sigma^{\mathrm{l}}(m_\Gamma^{\mathrm{r}} \otimes \mathrm{id})$$

$$(2.12) \qquad (m \otimes \mathrm{id})(\mathrm{id} \otimes \sigma^{\mathrm{l}})(\sigma^{\mathrm{l}} \otimes \mathrm{id}) = \sigma^{\mathrm{l}}(\mathrm{id} \otimes m)$$

(ii) Similarly, if Γ is right σ-covariant then

$$(2.13) \qquad \sigma^{\mathrm{r}}(1 \otimes \vartheta) = \vartheta \otimes 1 \qquad \sigma^{\mathrm{r}}(\mathrm{id} \otimes d) = (d \otimes \mathrm{id})\sigma$$

$$(2.14) \qquad (\mathrm{id} \otimes \sigma)(\sigma^{\mathrm{r}} \otimes \mathrm{id})(\mathrm{id} \otimes \sigma^{\mathrm{r}}) = (\sigma^{\mathrm{r}} \otimes \mathrm{id})(\mathrm{id} \otimes \sigma^{\mathrm{r}})(\sigma \otimes \mathrm{id}).$$

The map σ^{r} is surjective, its kernel is an $\mathcal{A}$-subbimodule of $\mathcal{A} \otimes \Gamma$ and the following identities hold

$$(2.15) \qquad (\mathrm{id} \otimes m)(\sigma^{\mathrm{r}} \otimes \mathrm{id})(\mathrm{id} \otimes \sigma^{\mathrm{r}}) = \sigma^{\mathrm{r}}(m \otimes \mathrm{id})$$

$$(2.16) \qquad (m_\Gamma^{\mathrm{r}} \otimes \mathrm{id})(\mathrm{id} \otimes \sigma)(\sigma^{\mathrm{r}} \otimes \mathrm{id}) = \sigma^{\mathrm{r}}(\mathrm{id} \otimes m_\Gamma^{\mathrm{r}})$$

$$(2.17) \qquad (m_\Gamma^{\mathrm{l}} \otimes \mathrm{id})(\mathrm{id} \otimes \sigma^{\mathrm{r}})(\sigma \otimes \mathrm{id}) = \sigma^{\mathrm{r}}(\mathrm{id} \otimes m_\Gamma^{\mathrm{l}})$$

(iii) *Finally, if Γ is bi-σ-covariant then*

$$(2.18) \qquad (\mathrm{id} \otimes \sigma^{\mathrm{r}})(\sigma \otimes \mathrm{id})(\mathrm{id} \otimes \sigma^{\mathrm{l}}) = (\sigma^{\mathrm{l}} \otimes \mathrm{id})(\mathrm{id} \otimes \sigma)(\sigma^{\mathrm{r}} \otimes \mathrm{id}).$$

Proof. Let us assume that Γ is left σ-covariant. Identities (2.8) are obvious. Let us check (2.10)–(2.12). Using (2.4) we obtain

$$
\begin{aligned}
\sigma^{\mathrm{l}}(\iota_\Gamma^{\mathrm{l}} \otimes m) &= (\mathrm{id} \otimes \iota_\Gamma^{\mathrm{l}})(\sigma \otimes \mathrm{id})(\mathrm{id} \otimes \sigma)(\mathrm{id}^2 \otimes m) \\
&= (\mathrm{id} \otimes \iota_\Gamma^{\mathrm{l}})(\sigma \otimes \mathrm{id})(\mathrm{id} \otimes m \otimes \mathrm{id})(\mathrm{id}^2 \otimes \sigma)(\mathrm{id} \otimes \sigma \otimes \mathrm{id}) \\
&= (m \otimes \iota_\Gamma^{\mathrm{l}})(\mathrm{id} \otimes \sigma \otimes \mathrm{id})(\sigma \otimes \sigma)(\mathrm{id} \otimes \sigma \otimes \mathrm{id}) \\
&= (m \otimes \mathrm{id})(\mathrm{id} \otimes \sigma^{\mathrm{l}})(\mathrm{id} \otimes \iota_\Gamma^{\mathrm{l}} \otimes \mathrm{id})(\sigma \otimes \mathrm{id}^2)(\mathrm{id} \otimes \sigma \otimes \mathrm{id}) \\
&= (m \otimes \mathrm{id})(\mathrm{id} \otimes \sigma^{\mathrm{l}})(\sigma^{\mathrm{l}} \otimes \mathrm{id})(\iota_\Gamma^{\mathrm{l}} \otimes \mathrm{id}^2).
\end{aligned}
$$

Furthermore,

$$
\begin{aligned}
\sigma^{\mathrm{l}}(m_\Gamma^{\mathrm{l}} \otimes \mathrm{id})(\mathrm{id} \otimes \iota_\Gamma^{\mathrm{l}} \otimes \mathrm{id}) &= \sigma^{\mathrm{l}}(\iota_\Gamma^{\mathrm{l}} \otimes \mathrm{id})(m \otimes \mathrm{id}^2) = (\mathrm{id} \otimes \iota_\Gamma^{\mathrm{l}})(\sigma \otimes \mathrm{id})(m \otimes \sigma) \\
&= (\mathrm{id} \otimes \iota_\Gamma^{\mathrm{l}})(\mathrm{id} \otimes m \otimes \mathrm{id})(\sigma \otimes \mathrm{id}^2)(\mathrm{id} \otimes \sigma \otimes \mathrm{id})(\mathrm{id}^2 \otimes \sigma) \\
&= (\mathrm{id} \otimes m_\Gamma^{\mathrm{l}})(\sigma \otimes \mathrm{id})(\mathrm{id}^2 \otimes \iota_\Gamma^{\mathrm{l}})(\mathrm{id} \otimes \sigma \otimes \mathrm{id})(\mathrm{id}^2 \otimes \sigma) \\
&= (\mathrm{id} \otimes m_\Gamma^{\mathrm{l}})(\sigma \otimes \mathrm{id})(\mathrm{id} \otimes \sigma^{\mathrm{l}})(\mathrm{id} \otimes \iota_\Gamma^{\mathrm{l}} \otimes \mathrm{id}).
\end{aligned}
$$

Similarly, we find

$$
\sigma^{\mathrm{l}}(m_\Gamma^{\mathrm{r}} \otimes \mathrm{id})(\iota_\Gamma^{\mathrm{l}} \otimes \mathrm{id}^2) = \sigma^{\mathrm{l}}\Big\{\iota_\Gamma^{\mathrm{l}}(\mathrm{id} \otimes m) \otimes \mathrm{id}\Big\} - \sigma^{\mathrm{l}}\Big\{\iota_\Gamma^{\mathrm{l}}(m \otimes \mathrm{id}) \otimes \mathrm{id}\Big\}.
$$

The first term on the right-hand side of the above equality is equal to

$$
(\mathrm{id} \otimes \iota_\Gamma^{\mathrm{l}})(\sigma \otimes \mathrm{id})(\mathrm{id} \otimes \sigma)(\mathrm{id} \otimes m \otimes \mathrm{id}) = \big(\mathrm{id} \otimes \iota_\Gamma^{\mathrm{l}}(\mathrm{id} \otimes m)\big)(\sigma \otimes \mathrm{id}^2)(\mathrm{id} \otimes \sigma \otimes \mathrm{id})(\mathrm{id}^2 \otimes \sigma),
$$

while the second term reads

$$
(\mathrm{id} \otimes \iota_\Gamma^{\mathrm{l}})(\sigma \otimes \mathrm{id})(\mathrm{id} \otimes \sigma)(m \otimes \mathrm{id}^2) = (\mathrm{id} \otimes \iota_\Gamma^{\mathrm{l}})(\mathrm{id} \otimes m \otimes \mathrm{id})(\sigma \otimes \mathrm{id}^2)(\mathrm{id} \otimes \sigma \otimes \mathrm{id})(\mathrm{id}^2 \otimes \sigma).
$$

Combining the last two expressions we conclude

$$
\begin{aligned}
\sigma^{\mathrm{l}}(m_\Gamma^{\mathrm{r}} \otimes \mathrm{id})(\iota_\Gamma^{\mathrm{l}} \otimes \mathrm{id}^2) &= (\mathrm{id} \otimes m_\Gamma^{\mathrm{r}})(\mathrm{id} \otimes \iota_\Gamma^{\mathrm{l}} \otimes \mathrm{id})(\sigma \otimes \mathrm{id}^2)(\mathrm{id} \otimes \sigma \otimes \mathrm{id})(\mathrm{id}^2 \otimes \sigma) \\
&= (\mathrm{id} \otimes m_\Gamma^{\mathrm{r}})(\sigma^{\mathrm{l}} \otimes \mathrm{id})(\iota_\Gamma^{\mathrm{l}} \otimes \sigma).
\end{aligned}
$$

We prove (2.9). Direct transformations give

$$
\begin{aligned}
(\mathrm{id} \otimes \sigma^{\mathrm{l}})(\sigma^{\mathrm{l}} \otimes \mathrm{id})(\iota_\Gamma^{\mathrm{l}} \otimes \sigma) &= \\
&= (\mathrm{id} \otimes \sigma^{\mathrm{l}})(\mathrm{id} \otimes \iota_\Gamma^{\mathrm{l}} \otimes \mathrm{id})(\sigma \otimes \mathrm{id}^2)(\mathrm{id} \otimes \sigma \otimes \mathrm{id})(\mathrm{id}^2 \otimes \sigma) \\
&= (\mathrm{id}^2 \otimes \iota_\Gamma^{\mathrm{l}})(\mathrm{id} \otimes \sigma \otimes \mathrm{id})(\sigma \otimes \sigma)(\mathrm{id} \otimes \sigma \otimes \mathrm{id})(\mathrm{id}^2 \otimes \sigma) \\
&= (\mathrm{id}^2 \otimes \iota_\Gamma^{\mathrm{l}})(\mathrm{id} \otimes \sigma \otimes \mathrm{id})(\sigma \otimes \mathrm{id}^2)(\mathrm{id} \otimes \sigma \otimes \mathrm{id})(\mathrm{id}^2 \otimes \sigma)(\mathrm{id} \otimes \sigma \otimes \mathrm{id}) \\
&= (\mathrm{id}^2 \otimes \iota_\Gamma^{\mathrm{l}})(\sigma \otimes \mathrm{id}^2)(\mathrm{id} \otimes \sigma \otimes \mathrm{id})(\sigma \otimes \sigma)(\mathrm{id} \otimes \sigma \otimes \mathrm{id}) \\
&= (\sigma \otimes \mathrm{id})(\mathrm{id} \otimes \sigma^{\mathrm{l}})(\mathrm{id} \otimes \iota_\Gamma^{\mathrm{l}} \otimes \mathrm{id})(\sigma \otimes \mathrm{id}^2)(\mathrm{id} \otimes \sigma \otimes \mathrm{id}) \\
&= (\sigma \otimes \mathrm{id})(\mathrm{id} \otimes \sigma^{\mathrm{l}})(\sigma^{\mathrm{l}} \otimes \mathrm{id})(\iota_\Gamma^{\mathrm{l}} \otimes \mathrm{id}).
\end{aligned}
$$

To prove the surjectivity of σ^{l}, it is sufficient to check that the elements of the form $a \otimes bd(q)$ belong to $\mathrm{im}(\sigma^{\mathrm{l}})$. Let us define

$$\omega = (m_{\Gamma}^{\mathrm{l}} \otimes \mathrm{id})(\mathrm{id} \otimes d \otimes \mathrm{id})(\mathrm{id} \otimes \sigma^{-1})(\sigma^{-1} \otimes \mathrm{id})(a \otimes b \otimes q).$$

Using (2.8) and (2.10) we obtain

$$\sigma^{\mathrm{l}}(\omega) = (\mathrm{id} \otimes m_{\Gamma}^{\mathrm{l}})(\sigma \otimes \mathrm{id})(\mathrm{id} \otimes \sigma^{\mathrm{l}})(\mathrm{id} \otimes d \otimes \mathrm{id})(\mathrm{id} \otimes \sigma^{-1})(\sigma^{-1} \otimes \mathrm{id})(a \otimes b \otimes q) = a \otimes bd(q).$$

The fact that $\ker(\sigma^{\mathrm{l}})$ is an $\mathcal{A}$-subbimodule of $\Gamma \otimes \mathcal{A}$ directly follows from equalities (2.10) and (2.12).

In such a way we have shown (i). The right σ-covariance case can be treated in a similar manner. Finally, if Γ is bi-σ-covariant then

$$\begin{aligned}
(\mathrm{id} \otimes \sigma^{\mathrm{r}})&(\sigma \otimes \mathrm{id})(\mathrm{id} \otimes \sigma^{\mathrm{l}})(\mathrm{id} \otimes \iota_{\Gamma}^{\mathrm{l}} \otimes \mathrm{id}) = \\
=&(\mathrm{id} \otimes \sigma^{\mathrm{r}})(\sigma \otimes \mathrm{id})(\mathrm{id}^2 \otimes \iota_{\Gamma}^{\mathrm{l}})(\mathrm{id} \otimes \sigma \otimes \mathrm{id})(\mathrm{id}^2 \otimes \sigma) \\
=&(\mathrm{id} \otimes \iota_{\Gamma}^{\mathrm{l}} \otimes \mathrm{id})(\mathrm{id}^2 \otimes \sigma)(\mathrm{id} \otimes \sigma \otimes \mathrm{id})(\sigma \otimes \mathrm{id}^2)(\mathrm{id} \otimes \sigma \otimes \mathrm{id})(\mathrm{id}^2 \otimes \sigma) \\
=&(\mathrm{id} \otimes \iota_{\Gamma}^{\mathrm{l}} \otimes \mathrm{id})(\sigma \otimes \sigma)(\mathrm{id} \otimes \sigma \otimes \mathrm{id})(\sigma \otimes \sigma) \\
=&(\mathrm{id} \otimes \iota_{\Gamma}^{\mathrm{l}} \otimes \mathrm{id})(\sigma \otimes \mathrm{id}^2)(\mathrm{id} \otimes \sigma \otimes \mathrm{id})(\mathrm{id}^2 \otimes \sigma)(\mathrm{id} \otimes \sigma \otimes \mathrm{id})(\sigma \otimes \mathrm{id}^2) \\
=&(\sigma^{\mathrm{l}} \otimes \mathrm{id})(\iota_{\Gamma}^{\mathrm{l}} \otimes \sigma)(\mathrm{id} \otimes \sigma \otimes \mathrm{id})(\sigma \otimes \mathrm{id}^2) \\
=&(\sigma^{\mathrm{l}} \otimes \mathrm{id})(\mathrm{id} \otimes \sigma)(\sigma^{\mathrm{r}} \otimes \mathrm{id})(\mathrm{id} \otimes \iota_{\Gamma}^{\mathrm{l}} \otimes \mathrm{id}),
\end{aligned}$$

and this completes the proof. $\square$

It is easy to construct 'pathological' examples in which maps σ^{l} or σ^{r} are not injective. However, besides certain technical complications such a structure gives nothing essentially new. For this reason, we shall assume from this moment that every left/right σ-covariant calculus we are dealing with possesses bijective flip-over operator σ^{l} or σ^{r}. Modulo this assumption, left σ-covariance and right σ^{-1}-covariance are equivalent properties. In other words,

$$(2.19) \qquad\qquad (\sigma^{\mathrm{l}})^{-1} = (\sigma^{-1})^{\mathrm{r}} \qquad (\sigma^{\mathrm{r}})^{-1} = (\sigma^{-1})^{\mathrm{l}}.$$

Now, we shall generalize the previous consideration to situations in which, instead of one, a system of mutually compatible braided quantum space [3] structures on X appears.

Definition 2. Let us assume that $\mathcal{A}$ is equipped with a braid system $\mathcal{T}$. Then we shall say that X is a $\mathcal{T}$-braided quantum space.

Definition 3. A first order calculus Γ over a $\mathcal{T}$-braided quantum space X is called left/right/bi $\mathcal{T}$-covariant iff it is left/right/bi γ-covariant for each braiding $\gamma \in \mathcal{T}$.

As explained in [3]-Appendix, every braid system $\mathcal{T}$ can be naturally *completed*. The completed system $\mathcal{T}^*$ is defined as the minimal extension of $\mathcal{T}$, invariant under ternar operations of the form $\delta = \alpha\beta^{-1}\gamma$. Explicitly, $\mathcal{T}^*$ is the union of systems $\mathcal{T}_n$, where $\mathcal{T}_0 = \mathcal{T}$ and $\mathcal{T}_{n+1}$ is obtained from $\mathcal{T}_n$ by applying the above mentioned operations.

Proposition 2.2. *Let X be a $\mathcal{T}$-braided quantum space and Γ a first-order calculus over X.*

(i) If Γ is left $\mathcal{T}$-covariant then it is also left $\mathcal{T}^*$-covariant. We have

$$(2.20) \qquad (\alpha\beta^{-1}\gamma)^{\mathrm{l}} = \alpha^{\mathrm{l}}(\beta^{\mathrm{l}})^{-1}\gamma^{\mathrm{l}}$$

$$(2.21) \qquad (\mathrm{id}\otimes\alpha^{\mathrm{l}})(\beta^{\mathrm{l}}\otimes\mathrm{id})(\mathrm{id}\otimes\gamma) = (\gamma\otimes\mathrm{id})(\mathrm{id}\otimes\beta^{\mathrm{l}})(\alpha^{\mathrm{l}}\otimes\mathrm{id}),$$

for each $\alpha,\beta,\gamma \in \mathcal{T}^*$.

(ii) Similarly, if Γ is right $\mathcal{T}$-covariant then it is right $\mathcal{T}^*$-covariant, too. We have

$$(2.22) \qquad (\alpha\beta^{-1}\gamma)^{\mathrm{r}} = \alpha^{\mathrm{r}}(\beta^{\mathrm{r}})^{-1}\gamma^{\mathrm{r}}$$

$$(2.23) \qquad (\mathrm{id}\otimes\alpha)(\beta^{\mathrm{r}}\otimes\mathrm{id})(\mathrm{id}\otimes\gamma^{\mathrm{r}}) = (\gamma^{\mathrm{r}}\otimes\mathrm{id})(\mathrm{id}\otimes\beta^{\mathrm{r}})(\alpha\otimes\mathrm{id})$$

for each $\alpha,\beta,\gamma \in \mathcal{T}^*$.

(iii) If Γ is bi-$\mathcal{T}$-covariant, it is consequently also bi-$\mathcal{T}^*$-covariant and

$$(2.24) \qquad (\mathrm{id}\otimes\alpha^{\mathrm{r}})(\beta\otimes\mathrm{id})(\mathrm{id}\otimes\gamma^{\mathrm{l}}) = (\gamma^{\mathrm{l}}\otimes\mathrm{id})(\mathrm{id}\otimes\beta)(\alpha^{\mathrm{r}}\otimes\mathrm{id})$$

for each $\alpha,\beta,\gamma \in \mathcal{T}^*$.

Proof. Let us assume that Γ is left $\mathcal{T}$-covariant. Then,

$$\alpha^{\mathrm{l}}(\beta^{\mathrm{l}})^{-1}\gamma^{\mathrm{l}}(\iota_\Gamma^{\mathrm{l}}\otimes\mathrm{id}) = (\mathrm{id}\otimes\iota_\Gamma^{\mathrm{l}})(\alpha\otimes\mathrm{id})(\mathrm{id}\otimes\alpha\beta^{-1})(\beta^{-1}\gamma\otimes\mathrm{id})(\mathrm{id}\otimes\gamma)$$
$$= (\mathrm{id}\otimes\iota_\Gamma^{\mathrm{l}})(\alpha\beta^{-1}\gamma\otimes\mathrm{id})(\mathrm{id}\otimes\alpha\beta^{-1}\gamma),$$

for each $\alpha,\beta,\gamma \in \mathcal{T}$. This means that Γ is left $\alpha\beta^{-1}\gamma$-covariant and (2.20) holds for the braidings from $\mathcal{T}$. Now, we can proceed inductively and conclude that Γ is left $\mathcal{T}_n$-covariant for each $n \in \mathbb{N}$, and that (2.20) holds on $\mathcal{T}^*$.

Similarly, if Γ is right $\mathcal{T}$-covariant then

$$\alpha^{\mathrm{r}}(\beta^{\mathrm{r}})^{-1}\gamma^{\mathrm{r}}(\mathrm{id}\otimes\iota_\Gamma^{\mathrm{r}}) = (\iota_\Gamma^{\mathrm{r}}\otimes\mathrm{id})(\mathrm{id}\otimes\alpha)(\alpha\beta^{-1}\otimes\mathrm{id})(\mathrm{id}\otimes\beta^{-1}\gamma)(\gamma\otimes\mathrm{id})$$
$$= (\iota_\Gamma^{\mathrm{r}}\otimes\mathrm{id})(\mathrm{id}\otimes\alpha\beta^{-1}\gamma)(\alpha\beta^{-1}\gamma\otimes\mathrm{id}),$$

for each $\alpha,\beta,\gamma \in \mathcal{T}$. This implies that Γ is right $\mathcal{T}^*$-covariant and that (2.22) holds for each $\alpha,\beta,\gamma \in \mathcal{T}^*$.

Identities (2.21) and (2.23)–(2.24) can be derived in essentially the same manner as it is done in the proof of Proposition 2.1, in the case of a single flip-over operator. $\square$

From this moment, as well as through the next three sections we shall deal exclusively with braided quantum groups, in the sense of [3]. Let G be such a group, represented by $\mathcal{A}$. We shall denote by $\phi\colon \mathcal{A} \to \mathcal{A}\otimes\mathcal{A}$ the coproduct map, and by $\epsilon\colon \mathcal{A} \to \mathbb{C}$ and $\kappa\colon \mathcal{A} \to \mathcal{A}$ the counit and the antipode map respectively. Let $\sigma\colon \mathcal{A}\otimes\mathcal{A} \to \mathcal{A}\otimes\mathcal{A}$ be the intrinsic braid operator.

As explained in [3], twisting properties of the coproduct and the antipode are not properly expressible in terms a single braid operator σ. This is the place where a 'secondary' braid operator naturally enters the game. Explicitly, it is given by

$$(2.25) \qquad \tau = (\epsilon\otimes\mathrm{id}^2)(\sigma^{-1}\otimes\mathrm{id})(\mathrm{id}\otimes\phi)\sigma = (\mathrm{id}^2\otimes\epsilon)(\mathrm{id}\otimes\sigma^{-1})(\phi\otimes\mathrm{id})\sigma.$$

The operators $\{\sigma, \tau\}$ form a braid system, and the completion $\mathcal{F} = \{\sigma, \tau\}^*$ is consisting of maps of the form

$$\sigma_n = (\sigma\tau^{-1})^{n-1}\sigma = \sigma(\tau^{-1}\sigma)^{n-1},$$

where $n \in \mathbb{Z}$.

Proposition 2.3. (i) *If Γ is left σ-covariant then it is also left $\mathcal{F}$-covariant and*

$$(2.26) \qquad \tau^{\mathrm{l}} = (\mathrm{id}^2 \otimes \epsilon)(\mathrm{id} \otimes (\sigma^{\mathrm{l}})^{-1})(\phi \otimes \mathrm{id})\sigma^{\mathrm{l}}$$

$$(2.27) \qquad (\tau^{\mathrm{l}})^{-1} = (\epsilon \otimes \mathrm{id}^2)(\sigma^{\mathrm{l}} \otimes \mathrm{id})(\mathrm{id} \otimes \phi)(\sigma^{\mathrm{l}})^{-1}$$

$$(2.28) \qquad (\epsilon \otimes \mathrm{id})\tau^{\mathrm{l}} = \mathrm{id} \otimes \epsilon.$$

Moreover, the following twisting properties hold

$$(2.29) \qquad (\mathrm{id} \otimes \sigma_n^{\mathrm{l}})(\sigma_m^{\mathrm{l}} \otimes \mathrm{id})(\mathrm{id} \otimes \phi) = (\phi \otimes \mathrm{id})\sigma_{m+n}^{\mathrm{l}}$$

for each $n, m \in \mathbb{Z}$.

$\quad$ (ii) *Similarly, if Γ is right σ-covariant then it is also right $\mathcal{F}$-covariant and*

$$(2.30) \qquad \tau^{\mathrm{r}} = (\epsilon \otimes \mathrm{id}^2)((\sigma^{\mathrm{r}})^{-1} \otimes \mathrm{id})(\mathrm{id} \otimes \phi)\sigma^{\mathrm{r}}$$

$$(2.31) \qquad (\tau^{\mathrm{r}})^{-1} = (\mathrm{id}^2 \otimes \epsilon)(\mathrm{id} \otimes \sigma^{\mathrm{r}})(\phi \otimes \mathrm{id})(\sigma^{\mathrm{r}})^{-1}$$

$$(2.32) \qquad (\mathrm{id} \otimes \epsilon)\tau^{\mathrm{r}} = \epsilon \otimes \mathrm{id}.$$

We also have

$$(2.33) \qquad (\sigma_n^{\mathrm{r}} \otimes \mathrm{id})(\mathrm{id} \otimes \sigma_m^{\mathrm{r}})(\phi \otimes \mathrm{id}) = (\mathrm{id} \otimes \phi)\sigma_{n+m}^{\mathrm{r}},$$

for each $n, m \in \mathbb{Z}$.

Proof. Let us assume left σ-covariance of Γ, and consider a map $\xi \colon \Gamma \otimes \mathcal{A} \to \mathcal{A} \otimes \Gamma$ determined by the right hand side of (2.26). Direct transformations give

$$\begin{aligned}
\xi(\iota_\Gamma^{\mathrm{l}} \otimes \mathrm{id}) &= (\mathrm{id}^2 \otimes \epsilon)(\mathrm{id} \otimes (\sigma^{\mathrm{l}})^{-1})(\phi \otimes \mathrm{id})\sigma^{\mathrm{l}}(\iota_\Gamma^{\mathrm{l}} \otimes \mathrm{id}) \\
&= (\mathrm{id} \otimes \iota_\Gamma^{\mathrm{l}} \otimes \epsilon)(\mathrm{id}^2 \otimes \sigma^{-1})(\mathrm{id} \otimes \sigma^{-1} \otimes \mathrm{id})(\phi \otimes \mathrm{id}^2)(\sigma \otimes \mathrm{id})(\mathrm{id} \otimes \sigma) \\
&= (\mathrm{id} \otimes \iota_\Gamma^{\mathrm{l}} \otimes \epsilon)(\tau \otimes \sigma^{-1})(\mathrm{id} \otimes \phi \otimes \mathrm{id})(\mathrm{id} \otimes \sigma) \\
&= (\mathrm{id} \otimes \iota_\Gamma^{\mathrm{l}})(\tau \otimes \mathrm{id})(\mathrm{id} \otimes \tau).
\end{aligned}$$

Consequently, Γ is left τ-covariant and $\xi = \tau^{\mathrm{l}}$. According to Proposition 2.2 the calculus is left $\mathcal{F}$-covariant.

Let us denote by ψ a map determined by the right hand side of (2.27). We have then

$$\begin{aligned}
\psi(\mathrm{id} \otimes \iota_\Gamma^{\mathrm{r}}) &= (\epsilon \otimes \mathrm{id}^2)(\sigma^{\mathrm{l}} \otimes \mathrm{id})(\mathrm{id} \otimes \phi)(\sigma^{\mathrm{l}})^{-1}(\mathrm{id} \otimes \iota_\Gamma^{\mathrm{r}}) \\
&= (\epsilon \otimes \iota_\Gamma^{\mathrm{r}} \otimes \mathrm{id})(\sigma \otimes \mathrm{id}^2)(\mathrm{id} \otimes \sigma \otimes \mathrm{id})(\mathrm{id}^2 \otimes \phi)(\mathrm{id} \otimes \sigma^{-1})(\sigma^{-1} \otimes \mathrm{id}) \\
&= (\epsilon \otimes \iota_\Gamma^{\mathrm{r}} \otimes \mathrm{id})(\sigma \otimes \tau^{-1})(\mathrm{id} \otimes \phi \otimes \mathrm{id})(\sigma^{-1} \otimes \mathrm{id}) \\
&= (\epsilon \otimes \iota_\Gamma^{\mathrm{r}} \otimes \mathrm{id})(\mathrm{id}^2 \otimes \tau^{-1})(\mathrm{id} \otimes \tau^{-1} \otimes \mathrm{id})(\phi \otimes \mathrm{id}) \\
&= (\iota_\Gamma^{\mathrm{r}} \otimes \mathrm{id})(\mathrm{id} \otimes \tau^{-1})(\tau^{-1} \otimes \mathrm{id}).
\end{aligned}$$

Consequently, Γ is right τ^{-1}-covariant and $\psi = (\tau^{-1})^{\mathrm{r}} = (\tau^{\mathrm{l}})^{-1}$.

Let us prove the twisting property (2.29). Using the standard braid relations we obtain

$$(\mathrm{id} \otimes \sigma_n^{\mathrm{l}})(\sigma_m^{\mathrm{l}} \otimes \mathrm{id})(\mathrm{id} \otimes \phi)(\iota_\Gamma^{\mathrm{l}} \otimes \mathrm{id}) =$$
$$= (\mathrm{id} \otimes \sigma_n^{\mathrm{l}})(\mathrm{id} \otimes \iota_\Gamma^{\mathrm{l}} \otimes \mathrm{id})(\sigma_m \otimes \mathrm{id}^2)(\mathrm{id} \otimes \sigma_m \otimes \mathrm{id})(\mathrm{id}^2 \otimes \phi)$$
$$= (\mathrm{id}^2 \otimes \iota_\Gamma^{\mathrm{l}})(\mathrm{id} \otimes \sigma_n \otimes \mathrm{id})(\sigma_m \otimes \sigma_n)(\mathrm{id} \otimes \sigma_m \otimes \mathrm{id})(\mathrm{id}^2 \otimes \phi)$$
$$= (\mathrm{id}^2 \otimes \iota_\Gamma^{\mathrm{l}})(\mathrm{id} \otimes \sigma_n \otimes \mathrm{id})(\sigma_m \otimes \mathrm{id}^2)(\mathrm{id} \otimes \phi \otimes \mathrm{id})(\mathrm{id} \otimes \sigma_{m+n})$$
$$= (\phi \otimes \iota_\Gamma^{\mathrm{l}})(\sigma_{n+m} \otimes \mathrm{id})(\mathrm{id} \otimes \sigma_{n+m}) = (\phi \otimes \mathrm{id})\sigma_{n+m}^{\mathrm{l}}(\iota_\Gamma^{\mathrm{l}} \otimes \mathrm{id}).$$

The case (ii), when Γ is right σ-covariant, can be treated in a similar way. $\square$

Finally, let us describe twisting relations between the antipode κ and a σ-covariant calculus Γ.

Proposition 2.4. *If Γ is left σ-covariant then*

$$(2.34) \qquad\qquad\qquad \sigma_n^{\mathrm{l}}(\mathrm{id} \otimes \kappa) = (\kappa \otimes \mathrm{id})\sigma_{-n}^{\mathrm{l}}.$$

Similarly, if Γ is right $\mathcal{F}$-covariant then

$$(2.35) \qquad\qquad\qquad \sigma_n^{\mathrm{r}}(\kappa \otimes \mathrm{id}) = (\mathrm{id} \otimes \kappa)\sigma_{-n}^{\mathrm{r}},$$

for each $n \in \mathbb{Z}$.

Proof. Let us assume that Γ is left σ-covariant. We have

$$\sigma^{\mathrm{l}}(\iota_\Gamma^{\mathrm{l}} \otimes \kappa) = (\mathrm{id} \otimes \iota_\Gamma^{\mathrm{l}})(\sigma_n \otimes \mathrm{id})(\mathrm{id} \otimes \sigma_n)(\mathrm{id} \otimes \kappa) = (\kappa \otimes \iota_\Gamma^{\mathrm{l}})(\sigma_{-n} \otimes \mathrm{id})(\mathrm{id} \otimes \sigma_{-n})$$
$$= (\kappa \otimes \mathrm{id})\sigma_{-n}^{\mathrm{l}}(\iota_\Gamma^{\mathrm{l}} \otimes \mathrm{id}).$$

If the calculus is right $\mathcal{F}$-covariant then

$$\sigma_n^{\mathrm{r}}(\kappa \otimes \iota_\Gamma^{\mathrm{r}}) = (\iota_\Gamma^{\mathrm{r}} \otimes \mathrm{id})(\mathrm{id} \otimes \sigma_n)(\sigma_n \otimes \mathrm{id})(\kappa \otimes \mathrm{id}^2) = (\iota_\Gamma^{\mathrm{r}} \otimes \kappa)(\mathrm{id} \otimes \sigma_{-n})(\sigma_{-n} \otimes \mathrm{id})$$
$$= (\mathrm{id} \otimes \kappa)\sigma_{-n}^{\mathrm{r}}(\mathrm{id} \otimes \iota_\Gamma^{\mathrm{r}}),$$

for each $n \in \mathbb{Z}$. $\square$

3. The Structure of Left-Covariant Calculi

We pass to definitions of first order differential structures which are covariant with respect to the comultiplication map $\phi \colon \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$.

Definition 4. A first-order calculus Γ over G is called right-covariant iff there exists a linear map $\varrho_\Gamma \colon \Gamma \to \Gamma \otimes \mathcal{A}$ such that

$$(3.1) \qquad\qquad\qquad \varrho_\Gamma \iota_\Gamma^{\mathrm{r}} = (\iota_\Gamma^{\mathrm{r}} \otimes m)(\mathrm{id} \otimes \sigma \otimes \mathrm{id})(\phi \otimes \phi).$$

The map ϱ_Γ is called the right action of G on Γ. It is uniquely determined by the above condition.

Definition 5. The calculus Γ is called left-covariant iff there exists a left action map $\ell_\Gamma \colon \Gamma \to \mathcal{A} \otimes \Gamma$ satisfying

$$(3.2) \qquad \ell_\Gamma \iota_\Gamma^{\mathrm{l}} = (m \otimes \iota_\Gamma^{\mathrm{l}})(\mathrm{id} \otimes \sigma \otimes \mathrm{id})(\phi \otimes \phi).$$

The map ℓ_Γ is uniquely determined by this condition.

Definition 6. We shall say that the calculus Γ is bicovariant, iff it is both left and right-covariant.

The above definitions naturally formulate braided generalizations of standard concepts of right/left and bi-covariance in the standard theory [5]. Throughout the rest of the section, we shall consider left-covariant differential structures.

Proposition 3.1. *We have*

$$(3.3) \qquad \ell_\Gamma d = (\mathrm{id} \otimes d)\phi$$

$$(3.4) \qquad \ell_\Gamma m_\Gamma^{\mathrm{l}} = (m \otimes m_\Gamma^{\mathrm{l}})(\mathrm{id} \otimes \sigma \otimes \mathrm{id})(\phi \otimes \ell_\Gamma).$$

Proof. Identity (3.3) is a direct consequence of (3.2). To prove (3.4), we start from (3.2) and apply elementary properties of the product and the coproduct maps:

$$
\begin{aligned}
(\ell_\Gamma m_\Gamma^{\mathrm{l}})(\mathrm{id} \otimes \iota_\Gamma^{\mathrm{l}}) &= \ell_\Gamma \iota_\Gamma^{\mathrm{l}}(m \otimes \mathrm{id}) = (m \otimes \iota_\Gamma^{\mathrm{l}})(\mathrm{id} \otimes \sigma \otimes \mathrm{id})(\phi \otimes \phi)(m \otimes \mathrm{id}) \\
&= (m \otimes \iota_\Gamma^{\mathrm{l}})(m \otimes \sigma \otimes \mathrm{id})(\mathrm{id}^2 \otimes m \otimes \mathrm{id}^2)(\mathrm{id} \otimes \sigma \otimes \mathrm{id}^3)(\phi \otimes \phi \otimes \phi) \\
&= (m \otimes \iota_\Gamma^{\mathrm{l}})(m \otimes \mathrm{id} \otimes m \otimes \mathrm{id})(\mathrm{id}^2 \otimes \sigma \otimes \mathrm{id}^2)(\mathrm{id} \otimes \sigma \otimes \sigma \otimes \mathrm{id})(\phi \otimes \phi \otimes \phi) \\
&= (m \otimes \iota_\Gamma^{\mathrm{l}})(\mathrm{id}^2 \otimes m \otimes \mathrm{id})\big[\mathrm{id} \otimes \sigma(\mathrm{id} \otimes m) \otimes \mathrm{id}^2\big](\mathrm{id}^3 \otimes \sigma \otimes \mathrm{id})(\phi \otimes \phi \otimes \phi) \\
&= (m \otimes m_\Gamma^{\mathrm{l}})\big[\mathrm{id} \otimes \sigma(\mathrm{id} \otimes m) \otimes \iota_\Gamma^{\mathrm{l}}\big](\mathrm{id}^3 \otimes \sigma \otimes \mathrm{id})(\phi \otimes \phi \otimes \phi) \\
&= (m \otimes m_\Gamma^{\mathrm{l}})(\mathrm{id} \otimes \sigma \otimes \mathrm{id})(\phi \otimes \ell_\Gamma \iota_\Gamma^{\mathrm{l}}). \quad \square
\end{aligned}
$$

It is worth noticing that

$$(3.5) \qquad \ell_\Gamma \iota_\Gamma^{\mathrm{r}} = (m \otimes \iota_\Gamma^{\mathrm{r}})(\mathrm{id} \otimes \sigma \otimes \mathrm{id})(\phi \otimes \phi),$$

which also characterizes the map ℓ_Γ. The following proposition shows that ℓ_Γ gives a left $\mathcal{A}$-comodule structure on Γ.

Proposition 3.2. *We have*

$$(3.6) \qquad (\epsilon \otimes \mathrm{id})\ell_\Gamma = \mathrm{id}$$

$$(3.7) \qquad (\phi \otimes \mathrm{id})\ell_\Gamma = (\mathrm{id} \otimes \ell_\Gamma)\ell_\Gamma.$$

Proof. Applying (3.2) and performing further elementary transormations with the counit we obtain

$$
\begin{aligned}
(\epsilon \otimes \mathrm{id})\ell_\Gamma \iota_\Gamma^{\mathrm{l}} &= (\epsilon \otimes \mathrm{id})(m \otimes \iota_\Gamma^{\mathrm{l}})(\mathrm{id} \otimes \sigma \otimes \mathrm{id})(\phi \otimes \phi) \\
&= (\epsilon \otimes \epsilon \otimes \iota_\Gamma^{\mathrm{l}})(\sigma^{-1}\tau \otimes \mathrm{id}^2)(\mathrm{id} \otimes \sigma \otimes \mathrm{id})(\phi \otimes \phi) \\
&= (\epsilon \otimes \epsilon \otimes \iota_\Gamma^{\mathrm{l}})(\sigma^{-1} \otimes \mathrm{id}^2)(\mathrm{id} \otimes \phi \otimes \mathrm{id})(\sigma \otimes \mathrm{id})(\mathrm{id} \otimes \phi) \\
&= (\epsilon \otimes \epsilon \otimes \iota_\Gamma^{\mathrm{l}})(\mathrm{id} \otimes \tau \otimes \mathrm{id})(\phi \otimes \phi) = \iota_\Gamma^{\mathrm{l}}.
\end{aligned}
$$

Furthermore,

$$(\mathrm{id} \otimes \ell_\Gamma)\ell_\Gamma \iota_\Gamma^{\mathrm{l}} = (m \otimes m \otimes \iota_\Gamma^{\mathrm{l}})(\mathrm{id}^3 \otimes \sigma \otimes \mathrm{id})(\mathrm{id}^2 \otimes \phi \otimes \phi)(\mathrm{id} \otimes \sigma \otimes \mathrm{id})(\phi \otimes \phi)$$

$$= (m \otimes m \otimes \iota_\Gamma^{\mathrm{l}})(\mathrm{id}^3 \otimes \sigma \otimes \mathrm{id})(\mathrm{id}^2 \otimes \phi \otimes \mathrm{id}^2)(\mathrm{id} \otimes \sigma \otimes \mathrm{id}^2)(\phi \otimes (\mathrm{id} \otimes \phi)\phi)$$

$$= (m \otimes m \otimes \iota_\Gamma^{\mathrm{l}})(\mathrm{id}^3 \otimes \sigma \otimes \mathrm{id})(\mathrm{id}^2 \otimes \phi \otimes \mathrm{id}^2)(\mathrm{id} \otimes \sigma \otimes \mathrm{id}^2)(\phi \otimes \phi \otimes \mathrm{id})(\mathrm{id} \otimes \phi)$$

$$= (m \otimes m \otimes \iota_\Gamma^{\mathrm{l}})(\mathrm{id} \otimes \sigma \otimes \mathrm{id}^3)(\mathrm{id}^2 \otimes \phi \otimes \mathrm{id}^2)(\mathrm{id}^2 \otimes \sigma \otimes \mathrm{id})(\mathrm{id} \otimes \phi \otimes \mathrm{id}^2)(\phi \otimes \phi)$$

$$= (m \otimes m \otimes \iota_\Gamma^{\mathrm{l}})(\mathrm{id} \otimes \sigma \otimes \mathrm{id}^3)(\phi \otimes \phi \otimes \mathrm{id}^2)(\mathrm{id} \otimes \sigma \otimes \mathrm{id})(\phi \otimes \phi)$$

$$= (\phi m \otimes \iota_\Gamma^{\mathrm{l}})(\mathrm{id} \otimes \sigma \otimes \mathrm{id})(\phi \otimes \phi) = (\phi \otimes \mathrm{id})\ell_\Gamma \iota_\Gamma^{\mathrm{l}},$$

which completes the proof. We have used the 'octagonal' compatibility property between ϕ and σ. $\square$

As we shall now see, every left-covariant differential calculus Γ is left σ-covariant. According to Proposition 2.3, this means that Γ is left $\mathcal{F}$-covariant, too.

Proposition 3.3. (i) *The calculus Γ is, being left-covariant, also left $\mathcal{F}$-covariant.*

(ii) *The diagram*

(3.8)

$$
\begin{array}{ccc}
\mathcal{A} \otimes \Gamma \otimes \mathcal{A} \otimes \mathcal{A} & \xrightarrow{\ \kappa \otimes \ell_\Gamma m_\Gamma^{\mathrm{r}} \otimes \kappa\ } & \mathcal{A} \otimes \mathcal{A} \otimes \Gamma \otimes \mathcal{A} \\
{\scriptstyle \ell_\Gamma \otimes \phi} \uparrow & & \downarrow {\scriptstyle m \otimes m_\Gamma^{\mathrm{r}}} \\
\Gamma \otimes \mathcal{A} & \xrightarrow[\ \sigma^{\mathrm{l}}\ ]{} & \mathcal{A} \otimes \Gamma
\end{array}
$$

is commutative.

Proof. Let $\xi \colon \Gamma \otimes \mathcal{A} \to \mathcal{A} \otimes \Gamma$ be a map determined by

$$\xi = (m \otimes m_\Gamma^{\mathrm{r}})(\kappa \otimes \ell_\Gamma m_\Gamma^{\mathrm{r}} \otimes \kappa)(\ell_\Gamma \otimes \phi).$$

We shall prove that ξ satisfies a characteristic property for the flip-over operator σ^{l}. A direct computation gives

$$\xi(\iota_\Gamma^{\mathrm{r}} \otimes \mathrm{id}) = (m \otimes m_\Gamma^{\mathrm{r}})(\kappa \otimes \ell_\Gamma m_\Gamma^{\mathrm{r}} \otimes \kappa)(m \otimes \iota_\Gamma^{\mathrm{r}} \otimes \mathrm{id}^2)(\mathrm{id} \otimes \sigma \otimes \mathrm{id}^3)(\phi \otimes \phi \otimes \phi)$$

$$= (m \otimes m_\Gamma^{\mathrm{r}})(m \otimes \ell_\Gamma \iota_\Gamma^{\mathrm{r}} \otimes \kappa)(\kappa \otimes \kappa \otimes \mathrm{id} \otimes m \otimes \mathrm{id})(\sigma_{-2} \otimes \mathrm{id}^4)(\mathrm{id} \otimes \sigma \otimes \mathrm{id}^3)(\phi \otimes \phi \otimes \phi)$$

$$= (m \otimes m_\Gamma^{\mathrm{r}})(m \otimes \ell_\Gamma \iota_\Gamma^{\mathrm{r}} \otimes \mathrm{id})(\kappa \otimes \kappa \otimes \mathrm{id}^3)A$$

$$= (m \otimes m_\Gamma^{\mathrm{r}})(m \otimes m \otimes \iota_\Gamma^{\mathrm{r}} \otimes \mathrm{id})(\mathrm{id}^3 \otimes \sigma \otimes \mathrm{id}^2)(\kappa \otimes \kappa \otimes \phi \otimes \phi \otimes \mathrm{id})A$$

$$= (m \otimes \iota_\Gamma^{\mathrm{r}})(m \otimes \mathrm{id}^2 \otimes m)(\mathrm{id}^2 \otimes \sigma \otimes \mathrm{id}^2)\big[\kappa \otimes m(\kappa \otimes \mathrm{id})\phi \otimes \mathrm{id} \otimes \phi \otimes \mathrm{id}\big]A$$

$$= (m \otimes \iota_\Gamma^{\mathrm{r}})(m \otimes \sigma \otimes m)(\kappa \otimes 1\epsilon \otimes \mathrm{id} \otimes \phi \otimes \mathrm{id})A,$$

where we have introduced $A = (\mathrm{id} \otimes \phi \otimes \mathrm{id}^2)(\sigma_{-1} \otimes m \otimes \kappa)(\mathrm{id} \otimes \phi \otimes \phi)$.

The last term in the above sequence of transformations can be further written as follows:

$$(m \otimes \iota_\Gamma^{\mathrm{r}})(\kappa \otimes \sigma \otimes m)(\mathrm{id}^2 \otimes \phi \otimes \mathrm{id})(\sigma_{-1} \otimes m \otimes \kappa)(\mathrm{id} \otimes \phi \otimes \phi) =$$

$$= (m \otimes \iota_\Gamma^{\mathrm{r}})(\mathrm{id} \otimes \sigma \otimes m)(\sigma \otimes \phi \otimes \mathrm{id})(\mathrm{id} \otimes \kappa \otimes m \otimes \kappa)(\mathrm{id} \otimes \phi \otimes \phi)$$

$$= (\mathrm{id} \otimes \iota_\Gamma^{\mathrm{r}})(\sigma \otimes \mathrm{id})\big[\mathrm{id} \otimes (m \otimes m)(\kappa \otimes \phi m \otimes \kappa)(\phi \otimes \phi)\big] = (\mathrm{id} \otimes \iota_\Gamma^{\mathrm{r}})(\sigma \otimes \mathrm{id})(\mathrm{id} \otimes \sigma).$$

Thus, Γ is left σ-covariant, $\chi = \sigma^{\mathrm{l}}$ and diagram (3.8) is commutative. According to (i)-Proposition 2.3 the calculus Γ is automatically left $\mathcal{F}$-covariant. $\square$

The operator σ^{l} figures in the right multiplicativity law for the left action map.

Proposition 3.4. *The diagram*

(3.9)

$$
\begin{array}{ccc}
\mathcal{A} \otimes \Gamma \otimes \mathcal{A} \otimes \mathcal{A} & \xrightarrow{\ \mathrm{id} \otimes \sigma^{\mathrm{l}} \otimes \mathrm{id}\ } & \mathcal{A} \otimes \mathcal{A} \otimes \Gamma \otimes \mathcal{A} \\
{\scriptstyle \ell_\Gamma \otimes \phi} \Big\uparrow & & \Big\downarrow {\scriptstyle m \otimes m_\Gamma^{\mathrm{r}}} \\
\Gamma \otimes \mathcal{A} & \xrightarrow[\ \ell_\Gamma m_\Gamma^{\mathrm{r}}\ ]{} & \mathcal{A} \otimes \Gamma
\end{array}
$$

is commutative.

Proof. According to Proposition 3.2 and diagram (3.8),

$$
\begin{aligned}
(m \otimes m_\Gamma^{\mathrm{r}})(\mathrm{id} \otimes \sigma^{\mathrm{l}} \otimes \mathrm{id})(\ell_\Gamma \otimes \phi) = \\
= (m \otimes m_\Gamma^{\mathrm{r}})\big[\mathrm{id} \otimes (m \otimes m_\Gamma^{\mathrm{r}})(\kappa \otimes \ell_\Gamma m_\Gamma^{\mathrm{r}} \otimes \kappa)(\ell_\Gamma \otimes \phi) \otimes \mathrm{id}\big](\ell_\Gamma \otimes \phi) \\
= (m \otimes m_\Gamma^{\mathrm{r}})(m \otimes \mathrm{id}^2 \otimes m)(\mathrm{id} \otimes \kappa \otimes \ell_\Gamma m_\Gamma^{\mathrm{r}} \otimes \kappa \otimes \mathrm{id})(\phi \otimes \mathrm{id}^2 \otimes \phi)(\ell_\Gamma \otimes \phi) \\
= (m \otimes m_\Gamma^{\mathrm{r}})(1\epsilon \otimes \ell_\Gamma m_\Gamma^{\mathrm{r}} \otimes 1\epsilon)(\ell_\Gamma \otimes \phi) = \ell_\Gamma m_\Gamma^{\mathrm{r}}. \quad \square
\end{aligned}
$$

Our next proposition describes twisting properties of the left action map, with respect to the braid system $\mathcal{F}$.

Proposition 3.5. *We have*

(3.10)
$$
(\sigma_n \otimes \mathrm{id})(\mathrm{id} \otimes \sigma_m^{\mathrm{l}})(\ell_\Gamma \otimes \mathrm{id}) = (\mathrm{id} \otimes \ell_\Gamma)\sigma_{n+m}^{\mathrm{l}},
$$

for each $n, m \in \mathbb{Z}$. In particular, it follows that

(3.11)
$$
\tau^{\mathrm{l}} = (\epsilon \otimes \mathrm{id}^2)(\sigma^{-1} \otimes \mathrm{id})(\mathrm{id} \otimes \ell_\Gamma)\sigma^{\mathrm{l}}.
$$

Proof. Using (3.2) and the main properties of $\mathcal{F}$ we obtain

$$
\begin{aligned}
(\sigma_n \otimes \mathrm{id})(\mathrm{id} \otimes \sigma_m^{\mathrm{l}})(\ell_\Gamma \otimes \mathrm{id})(\iota_\Gamma^{\mathrm{l}} \otimes \mathrm{id}) = \\
= (\sigma_n \otimes \mathrm{id})(\mathrm{id} \otimes \sigma_m^{\mathrm{l}})(m \otimes \iota_\Gamma^{\mathrm{l}} \otimes \mathrm{id})(\mathrm{id} \otimes \sigma \otimes \mathrm{id}^2)(\phi \otimes \phi \otimes \mathrm{id}) \\
= (\sigma_n \otimes \mathrm{id})(m \otimes \mathrm{id} \otimes \iota_\Gamma^{\mathrm{l}})(\mathrm{id}^2 \otimes \sigma_m \otimes \mathrm{id})(\mathrm{id} \otimes \sigma \otimes \sigma_m)(\phi \otimes \phi \otimes \mathrm{id}) \\
= (\mathrm{id} \otimes m \otimes \iota_\Gamma^{\mathrm{l}})(\sigma_n \otimes \mathrm{id}^3)(\mathrm{id} \otimes \sigma_n \otimes \mathrm{id}^2)(\mathrm{id}^2 \otimes \sigma_m \otimes \mathrm{id})(\mathrm{id} \otimes \sigma \otimes \sigma_m)(\phi \otimes \phi \otimes \mathrm{id}) \\
= (\mathrm{id} \otimes m \otimes \iota_\Gamma^{\mathrm{l}})(\sigma_n \otimes \sigma \otimes \mathrm{id})(\mathrm{id} \otimes \sigma_m \otimes \mathrm{id}^2)(\mathrm{id}^2 \otimes \sigma_n \otimes \mathrm{id})(\mathrm{id}^3 \otimes \sigma_m)(\phi \otimes \phi \otimes \mathrm{id}) \\
= (\mathrm{id} \otimes m \otimes \iota_\Gamma^{\mathrm{l}})(\sigma_n \otimes \sigma \otimes \mathrm{id})(\mathrm{id} \otimes \sigma_m \otimes \mathrm{id}^2)(\mathrm{id}^3 \otimes \phi)(\phi \otimes \sigma_{n+m}) \\
= (\mathrm{id} \otimes m \otimes \iota_\Gamma^{\mathrm{l}})(\mathrm{id}^2 \otimes \sigma \otimes \mathrm{id})(\mathrm{id} \otimes \phi \otimes \phi)(\sigma_{n+m} \otimes \mathrm{id})(\mathrm{id} \otimes \sigma_{n+m}) \\
= (\mathrm{id} \otimes \ell_\Gamma \iota_\Gamma^{\mathrm{l}})(\sigma_{n+m} \otimes \mathrm{id})(\mathrm{id} \otimes \sigma_{n+m}) = (\mathrm{id} \otimes \ell_\Gamma)\sigma_{n+m}^{\mathrm{l}}(\iota_\Gamma^{\mathrm{l}} \otimes \mathrm{id}). \quad \square
\end{aligned}
$$

We pass to the study of the internal structure of left-covariant calculi. For a given Γ, let Γ_{inv} be the space of left-invariant elements of Γ. In other words

$$(3.12) \qquad \Gamma_{inv} = \Big\{ \vartheta \in \Gamma \mid \ell_\Gamma(\vartheta) = 1 \otimes \vartheta \Big\}.$$

Let $P \colon \Gamma \to \Gamma$ be a linear map defined by

$$(3.13) \qquad P = m_\Gamma^1(\kappa \otimes \mathrm{id})\ell_\Gamma.$$

We are going to show that P projects Γ onto Γ_{inv}. Evidently, the elements of Γ_{inv} are P-invariant.

Lemma 3.6. *We have*

$$(3.14) \qquad P\iota_\Gamma^1 = (\epsilon \otimes Pd)\sigma^{-1}\tau.$$

Proof. Applying (3.2)–(3.3), (3.13) and performing standard braided transformations we obtain

$$
\begin{aligned}
P\iota_\Gamma^1 &= m_\Gamma^1(\kappa \otimes \mathrm{id})(m \otimes \iota_\Gamma^1)(\mathrm{id} \otimes \sigma \otimes \mathrm{id})(\phi \otimes \phi) \\
&= m_\Gamma^1(m \otimes \iota_\Gamma^1)(\kappa \otimes \kappa \otimes \mathrm{id}^2)(\tau\sigma^{-1}\tau\sigma^{-1}\tau \otimes \mathrm{id}^2)(\mathrm{id} \otimes \sigma \otimes \mathrm{id})(\phi \otimes \phi) \\
&= m_\Gamma^1(m \otimes \iota_\Gamma^1)(k \otimes k \otimes \mathrm{id}^2)(\mathrm{id} \otimes \phi \otimes \mathrm{id})(\tau\sigma^{-1}\tau \otimes \mathrm{id})(\mathrm{id} \otimes \phi) \\
&= m_\Gamma^1(\mathrm{id} \otimes \iota_\Gamma^1)(\kappa \otimes 1\epsilon \otimes \mathrm{id})(\tau \otimes \mathrm{id})(\mathrm{id} \otimes \phi)\sigma^{-1}\tau \\
&= m_\Gamma^1(\epsilon \otimes \kappa \otimes d)(\mathrm{id} \otimes \phi)\sigma^{-1}\tau = (\epsilon \otimes Pd)\sigma^{-1}\tau. \quad \square
\end{aligned}
$$

Now, it follows that $P(\Gamma) \subseteq \Gamma_{inv}$. Indeed, according to the previous lemma, it is sufficient to check that $Pd(\mathcal{A}) \subseteq \Gamma_{inv}$. We compute

$$
\begin{aligned}
\ell_\Gamma Pd &= \ell_\Gamma m_\Gamma^1(\kappa \otimes d)\phi = (m \otimes m_\Gamma^1)(\mathrm{id} \otimes \sigma \otimes \mathrm{id})(\phi \otimes \ell_\Gamma)(\kappa \otimes d)\phi \\
&= (m \otimes m_\Gamma^1)(\mathrm{id} \otimes \sigma \otimes \mathrm{id})(\sigma \otimes \mathrm{id}^2)(\kappa \otimes \kappa \otimes \mathrm{id} \otimes d)(\phi \otimes \phi)\phi \\
&= (\mathrm{id} \otimes m_\Gamma^1)(\sigma \otimes \mathrm{id})(\mathrm{id} \otimes m \otimes \mathrm{id})(\kappa \otimes \kappa \otimes \mathrm{id} \otimes d)(\mathrm{id} \otimes \phi \otimes \mathrm{id})(\phi \otimes \mathrm{id})\phi \\
&= (\mathrm{id} \otimes m_\Gamma^1)(\sigma \otimes \mathrm{id})(\kappa \otimes 1\epsilon \otimes d)(\phi \otimes \mathrm{id})\phi \\
&= 1 \otimes m_\Gamma^1(\kappa \otimes d)\phi = 1 \otimes Pd.
\end{aligned}
$$

Consequently, P projects Γ onto Γ_{inv} and the composition

$$\pi = Pd = m_\Gamma^1(\kappa \otimes d)\phi \colon \mathcal{A} \to \Gamma_{inv}$$

is surjective.

It is easy to see, by the use of (3.10), that the flip-over operators σ_n^1 map $\Gamma_{inv} \otimes \mathcal{A}$ onto $\mathcal{A} \otimes \Gamma_{inv}$. Moreover, the corresponding restrictions mutually coincide.

Lemma 3.7. *We have*

$$(3.15) \qquad \sigma_n^1(\pi \otimes \mathrm{id}) = (\mathrm{id} \otimes \pi)\tau$$

for each $n \in \mathbb{Z}$.

Proof. Applying the appropriate twisting properties we obtain

$$\sigma_n^1(\pi \otimes \mathrm{id}) = \sigma_n^1(\iota_\Gamma^1 \otimes \mathrm{id})(\kappa \otimes \mathrm{id}^2)(\phi \otimes \mathrm{id})$$
$$= (\mathrm{id} \otimes \iota_\Gamma^1)(\sigma_n \otimes \mathrm{id})(\mathrm{id} \otimes \sigma_n)(\kappa \otimes \mathrm{id}^2)(\phi \otimes \mathrm{id})$$
$$= (\mathrm{id} \otimes \iota_\Gamma^1)(\mathrm{id} \otimes \kappa \otimes \mathrm{id})(\sigma_{-n} \otimes \mathrm{id})(\mathrm{id} \otimes \sigma_n)(\phi \otimes \mathrm{id})$$
$$= (\mathrm{id} \otimes \iota_\Gamma^1)(\mathrm{id} \otimes \kappa \otimes \mathrm{id})(\mathrm{id} \otimes \phi)\tau = (\mathrm{id} \otimes \pi)\tau. \quad \square$$

We are going to prove that the space Γ is naturally isomorphic to $\mathcal{A} \otimes \Gamma_{inv}$, as a left $\mathcal{A}$-module.

Proposition 3.8. *Let us consider the map*

$$(\mathrm{id} \otimes P)\ell_\Gamma \colon \Gamma \to \mathcal{A} \otimes \Gamma_{inv}.$$

This is an isomorphism of left $\mathcal{A}$-modules. Its inverse is given by

$$(m_\Gamma^1 {\restriction} \mathcal{A} \otimes \Gamma_{inv}) \colon \mathcal{A} \otimes \Gamma_{inv} \to \Gamma.$$

Proof. The map $m_\Gamma^1 {\restriction} \mathcal{A} \otimes \Gamma_{inv}$ is evidently a left $\mathcal{A}$-module homomorphism. Let us check that $m_\Gamma^1 {\restriction} \mathcal{A} \otimes \Gamma_{inv}$ and $(\mathrm{id} \otimes P)\ell_\Gamma$ are mutually inverse maps. Using (3.6)–(3.7) and (3.13) we obtain

$$m_\Gamma^1(\mathrm{id} \otimes P)\ell_\Gamma = m_\Gamma^1(\mathrm{id} \otimes m_\Gamma^1)(\mathrm{id} \otimes \kappa \otimes \mathrm{id})(\mathrm{id} \otimes \ell_\Gamma)\ell_\Gamma$$
$$= m_\Gamma^1(m \otimes \mathrm{id})(\mathrm{id} \otimes \kappa \otimes \mathrm{id})(\phi \otimes \mathrm{id})\ell_\Gamma = m_\Gamma^1(\epsilon 1 \otimes \mathrm{id})\ell_\Gamma = \mathrm{id}.$$

On the other hand

$$P(a\vartheta) = \epsilon(a)\vartheta$$

for each $a \in \mathcal{A}$ and $\vartheta \in \Gamma_{inv}$. Using this and (3.4) we obtain

$$(\mathrm{id} \otimes P)\ell_\Gamma(a\vartheta) = (\mathrm{id} \otimes P)(m \otimes m_\Gamma^1)(\mathrm{id} \otimes \sigma \otimes \mathrm{id})(\phi(a) \otimes 1 \otimes \vartheta)$$
$$= a^{(1)} \otimes P(a^{(2)}\vartheta) = a \otimes \vartheta.$$

Consequently, the two maps are mutually inverse left $\mathcal{A}$-module isomorphisms. $\quad \square$

The above proposition allows us to identify $\Gamma \leftrightarrow \mathcal{A} \otimes \Gamma_{inv}$. In terms of this identification, the following correspondences hold

(3.16) $$d \leftrightarrow (\mathrm{id} \otimes \pi)\phi$$
(3.17) $$\ell_\Gamma \leftrightarrow \phi \otimes \mathrm{id}$$
(3.18) $$m_\Gamma^1 \leftrightarrow m \otimes \mathrm{id}.$$

The following technical lemma will be useful in some further computations.

Lemma 3.9. *We have*

(3.19) $$P[\pi(a)b] = \pi m \tau^{-1}\sigma(a \otimes b) - \epsilon(a)\pi(b),$$

for each $a, b \in \mathcal{A}$.

Proof. We compute

$$Pm_\Gamma^{\mathrm{r}}(\pi \otimes \mathrm{id}) = m_\Gamma^{\mathrm{l}}(\kappa \otimes \mathrm{id})\ell_\Gamma m_\Gamma^{\mathrm{r}}(\pi \otimes \mathrm{id})$$
$$= m_\Gamma^{\mathrm{l}}(\kappa \otimes \mathrm{id})(m \otimes m_\Gamma^{\mathrm{r}})(\mathrm{id} \otimes \sigma^1 \otimes \mathrm{id})(1 \otimes \pi \otimes \phi).$$

According to (3.15), this is further equal to

$$m_\Gamma^{\mathrm{l}}(\kappa \otimes \mathrm{id})(m \otimes m_\Gamma^{\mathrm{r}})(1 \otimes \mathrm{id} \otimes \pi \otimes \mathrm{id})(\tau \otimes \mathrm{id})(\mathrm{id} \otimes \phi) =$$
$$= m_\Gamma^{\mathrm{l}}(\kappa \otimes \mathrm{id})(\mathrm{id} \otimes m_\Gamma^{\mathrm{r}})(\mathrm{id} \otimes m_\Gamma^{\mathrm{l}} \otimes \mathrm{id})(\mathrm{id} \otimes \kappa \otimes d \otimes \mathrm{id})(\mathrm{id} \otimes \phi \otimes \mathrm{id})(\tau \otimes \mathrm{id})(\mathrm{id} \otimes \phi)$$
$$= m_\Gamma^{\mathrm{l}}(\kappa \otimes \mathrm{id})(\mathrm{id} \otimes m_\Gamma^{\mathrm{l}})(\mathrm{id} \otimes \kappa \otimes dm)(\mathrm{id} \otimes \phi \otimes \mathrm{id})(\tau \otimes \mathrm{id})(\mathrm{id} \otimes \phi)$$
$$- m_\Gamma^{\mathrm{l}}(\kappa \otimes \mathrm{id})(\mathrm{id} \otimes m_\Gamma^{\mathrm{l}})(\mathrm{id}^2 \otimes m_\Gamma^{\mathrm{l}})(\mathrm{id} \otimes \kappa \otimes \mathrm{id} \otimes d)(\mathrm{id} \otimes \phi \otimes \mathrm{id})(\tau \otimes \mathrm{id})(\mathrm{id} \otimes \phi).$$

The first term in the above difference is transformed further

$$m_\Gamma^{\mathrm{l}}(m \otimes \mathrm{id})(\kappa \otimes \kappa \otimes dm)(\mathrm{id} \otimes \phi \otimes \mathrm{id})(\tau \otimes \mathrm{id})(\mathrm{id} \otimes \phi) =$$
$$= m_\Gamma^{\mathrm{l}}(\kappa m \otimes dm)(\tau^{-1}\sigma\tau^{-1}\sigma\tau^{-1} \otimes \mathrm{id}^2)(\mathrm{id} \otimes \phi \otimes \mathrm{id})(\tau \otimes \mathrm{id})(\mathrm{id} \otimes \phi)$$
$$= m_\Gamma^{\mathrm{l}}(\kappa m \otimes dm)(\tau^{-1} \otimes \mathrm{id}^2)(\mathrm{id} \otimes \phi \otimes \mathrm{id})(\sigma\tau^{-1}\sigma \otimes \mathrm{id})(\mathrm{id} \otimes \phi)$$
$$= m_\Gamma^{\mathrm{l}}(\kappa m \otimes dm)(\mathrm{id} \otimes \sigma \otimes \mathrm{id})(\phi \otimes \phi)\tau^{-1}\sigma = m_\Gamma^{\mathrm{l}}(\kappa \otimes d)\phi m \tau^{-1}\sigma = \pi m \tau^{-1}\sigma.$$

Concerning the second term,

$$m_\Gamma^{\mathrm{l}}(\kappa \otimes \mathrm{id})(\mathrm{id} \otimes m_\Gamma^{\mathrm{l}})(\mathrm{id} \otimes 1\epsilon \otimes d)(\tau \otimes \mathrm{id})(\mathrm{id} \otimes \phi) = m_\Gamma^{\mathrm{l}}(\kappa \otimes d)(\epsilon \otimes \phi) = \epsilon \otimes \pi,$$

which completes the proof. $\square$

Let $\mathcal{R}$ be the intersection of spaces $\ker(\pi)$ and $\ker(\epsilon)$. As follows directly from the previous lemma, the space $\mathcal{R}$ is a right ideal in the algebra $\mathcal{A}_0$, which coincides as a vector space with $\mathcal{A}$, but which is endowed with the product $m_0 = m\tau^{-1}\sigma$, as discussed in [3]-Appendix. According to (3.15), we have

$$(3.20) \qquad\qquad \tau(\mathcal{R} \otimes \mathcal{A}) = \mathcal{A} \otimes \mathcal{R}.$$

The map π induces the isomorphism

$$(3.21) \qquad\qquad \Gamma_{inv} \leftrightarrow \ker(\epsilon)/\mathcal{R}.$$

It is easy to see that the map $\circ\colon \Gamma_{inv} \otimes \mathcal{A} \to \Gamma_{inv}$ given by

$$(3.22) \qquad\qquad \pi(a) \circ b = P(\pi(a)b) = \pi m_0(a \otimes b)$$

defines a right $\mathcal{A}_0$-module structure on the space Γ_{inv}. In the above formula it is assumed that $a \in \ker(\epsilon)$, while b is arbitrary.

In terms of the identification $\Gamma \leftrightarrow \mathcal{A} \otimes \Gamma_{inv}$ the right $\mathcal{A}$-module structure is given by

$$m_\Gamma^{\mathrm{r}} \leftrightarrow (m \otimes \circ)(\mathrm{id} \otimes \sigma_* \otimes \mathrm{id})(\mathrm{id}^2 \otimes \phi),$$

where $\sigma_*\colon \Gamma_{inv} \otimes \mathcal{A} \to \mathcal{A} \otimes \Gamma_{inv}$ is the common left-invariant part of all operators σ_n^1. We shall now prove that Γ is trivial as a right $\mathcal{A}$-module.

Proposition 3.10. *The multiplication map*

$$(3.23) \qquad (m_\Gamma^{\mathrm{r}} {\restriction} \Gamma_{inv} \otimes \mathcal{A}) \colon \Gamma_{inv} \otimes \mathcal{A} \to \Gamma = \mathcal{A} \otimes \Gamma_{inv}$$

is an isomorphism of right $\mathcal{A}$-modules. Its inverse is given by

$$(3.24) \qquad (\circ \otimes \kappa)(\mathrm{id} \otimes \phi\kappa^{-1})\sigma_*^{-1} \colon \mathcal{A} \otimes \Gamma_{inv} \to \Gamma_{inv} \otimes \mathcal{A}.$$

Proof. Clearly, (3.23) is a right $\mathcal{A}$-module homomorphism. A direct computation gives

$$\big[(\mathrm{id} \otimes \circ)(\sigma_* \otimes \mathrm{id})(\circ \otimes \phi\kappa)(\mathrm{id} \otimes \phi\kappa^{-1})\sigma_*^{-1}\big](\mathrm{id} \otimes \pi) =$$
$$= (\mathrm{id} \otimes \circ)(\sigma_* \otimes \mathrm{id})(\circ \otimes \phi\kappa)(\pi \otimes \phi\kappa^{-1})\tau^{-1}$$
$$= (\mathrm{id} \otimes \circ)(\sigma_* \otimes \mathrm{id})(\pi m\tau^{-1}\sigma \otimes \phi\kappa)(\mathrm{id} \otimes \phi\kappa^{-1})\tau^{-1}$$
$$= (\mathrm{id} \otimes \pi)(\mathrm{id} \otimes m\tau^{-1}\sigma)(\tau \otimes \mathrm{id})(m \otimes \phi\kappa)(\mathrm{id} \otimes \phi)\tau^{-1}\sigma(\mathrm{id} \otimes \kappa^{-1})\tau^{-1}$$
$$= (\mathrm{id} \otimes \pi)(\mathrm{id} \otimes m\tau^{-1})(\phi \otimes \mathrm{id})\sigma(m \otimes \kappa)(\mathrm{id} \otimes \phi\kappa^{-1})\sigma^{-1}$$
$$= (\mathrm{id} \otimes \pi)(\mathrm{id} \otimes m)(\sigma \otimes \mathrm{id})\big[m \otimes \sigma(\kappa \otimes \kappa)\phi\big](\mathrm{id} \otimes \phi\kappa^{-1})\sigma^{-1}$$
$$= (\mathrm{id} \otimes \pi)\sigma(m \otimes \mathrm{id})(m \otimes \kappa \otimes \kappa)(\mathrm{id} \otimes \phi \otimes \mathrm{id})(\mathrm{id} \otimes \phi\kappa^{-1})\sigma^{-1}$$
$$= (\mathrm{id} \otimes \pi)\sigma(m \otimes \mathrm{id})(\mathrm{id} \otimes 1\epsilon \otimes \kappa)(\mathrm{id} \otimes \phi\kappa^{-1})\sigma^{-1} = (\mathrm{id} \otimes \pi).$$

Furthermore,

$$(\circ \otimes \kappa)(\mathrm{id} \otimes \phi\kappa^{-1})\sigma_*^{-1}(\mathrm{id} \otimes \circ)(\sigma_* \otimes \mathrm{id})(\pi \otimes \phi) =$$
$$= (\circ \otimes \kappa)(\mathrm{id} \otimes \phi\kappa^{-1})\sigma_*^{-1}(\mathrm{id} \otimes \pi m\tau^{-1}\sigma)(\tau \otimes \mathrm{id})(\mathrm{id} \otimes \phi)$$
$$= (\pi m\tau^{-1}\sigma \otimes \kappa)(\mathrm{id} \otimes \phi\kappa^{-1})\tau^{-1}(\mathrm{id} \otimes m\tau^{-1}\sigma)(\tau \otimes \mathrm{id})(\mathrm{id} \otimes \phi)$$
$$= (\pi m \otimes \kappa)(\mathrm{id} \otimes \phi\kappa^{-1})\sigma^{-1}(\mathrm{id} \otimes m)(\sigma \otimes \mathrm{id})(\mathrm{id} \otimes \phi)$$
$$= (\pi m \otimes \kappa)(m \otimes \phi\kappa^{-1})(\mathrm{id} \otimes \sigma^{-1})(\mathrm{id} \otimes \phi)$$
$$= (\pi m \otimes \mathrm{id})(\mathrm{id} \otimes m \otimes \mathrm{id})(\mathrm{id} \otimes \kappa \otimes \mathrm{id} \otimes \kappa)(\mathrm{id} \otimes \phi \otimes \mathrm{id})(\mathrm{id} \otimes \phi\kappa^{-1})$$
$$= (\pi m \otimes \mathrm{id})(\mathrm{id} \otimes 1\epsilon \otimes \kappa)(\mathrm{id} \otimes \phi\kappa^{-1}) = (\pi \otimes \mathrm{id}).$$

The above computations are performed in the spaces $\mathcal{A} \otimes \ker(\epsilon)$ and $\ker(\epsilon) \otimes \mathcal{A}$ respectively. $\square$

In the framework of the identification $\Gamma \leftrightarrow \Gamma_{inv} \otimes \mathcal{A}$, the following correspondences hold:

$$(3.25) \qquad -d \leftrightarrow (\pi\kappa^{-1} \otimes \mathrm{id})\sigma^{-1}\phi = (\pi \otimes \kappa)\phi\kappa^{-1}$$
$$(3.26) \qquad m_\Gamma^{\mathrm{r}} \leftrightarrow \mathrm{id} \otimes m$$
$$(3.27) \qquad \ell_\Gamma \leftrightarrow (\sigma_* \otimes \mathrm{id})(\mathrm{id} \otimes \phi)$$
$$(3.28) \qquad m_\Gamma^{\mathrm{l}} \leftrightarrow \big[\circ(\mathrm{id} \otimes \kappa^{-1}) \otimes m\big](\mathrm{id} \otimes \sigma^{-1}\phi \otimes \mathrm{id})(\sigma_*^{-1} \otimes \mathrm{id}).$$

These correspondences follow from (3.23)–(3.24), performing simple algebraic transformations.

We are ready to present a braided counterpart of the reconstruction theorem [5] of the standard theory. As we have seen, every left-covariant calculus Γ is completely determined by the corresponding $\mathcal{R}$. The following proposition shows

that conversely, every right $\mathcal{A}_0$-ideal which satisfies (3.20) naturally gives rise to a first-order left-covariant calculus.

Proposition 3.11. *Let $\mathcal{R} \subseteq \ker(\epsilon)$ be an arbitrary τ-invariant right $\mathcal{A}_0$-ideal. Let us define spaces Γ_{inv} and Γ, together with maps $\circ\colon \Gamma_{inv} \otimes \mathcal{A} \to \Gamma_{inv}$ and $\pi\colon \mathcal{A} \to \Gamma_{inv}$, as well as $\sigma_*\colon \Gamma_{inv} \otimes \mathcal{A} \to \mathcal{A} \otimes \Gamma_{inv}$ by the equalities*

$$\Gamma_{inv} = \ker(\epsilon)/\mathcal{R} \qquad \Gamma = \mathcal{A} \otimes \Gamma_{inv}$$

$$\pi(a) = \big[a - \epsilon(a)\big]_{\mathcal{R}}$$

$$(\pi(a) \circ b) = \pi\big[m_0(a \otimes b) - \epsilon(a)b\big]$$

$$\sigma_*(\pi \otimes \mathrm{id}) = (\mathrm{id} \otimes \pi)\tau.$$

Finally, let us define maps

$$\ell_\Gamma\colon \Gamma \to \mathcal{A} \otimes \Gamma \qquad d\colon \mathcal{A} \to \Gamma \qquad m_\Gamma^{\mathrm{l}}\colon \mathcal{A} \otimes \Gamma \to \Gamma \qquad m_\Gamma^{\mathrm{r}}\colon \Gamma \otimes \mathcal{A} \to \Gamma$$

by equalities (3.17)–(3.18) and (3.23) respectively.

Then, m_Γ^{l} and m_Γ^{r} determine a structure of a unital $\mathcal{A}$-bimodule on Γ. Moreover, Γ is a left-covariant first-order differential calculus over G, with the differential and the left action coinciding with the introduced d and ℓ_Γ respectively.

Proof. It is clear that m_Γ^{l} determines a left $\mathcal{A}$-module structure on Γ. Let us prove that m_Γ^{r} determines a right $\mathcal{A}$-module structure. We have

$$m_\Gamma^{\mathrm{r}}(m_\Gamma^{\mathrm{r}} \otimes \mathrm{id}) = (m \otimes \circ)(\mathrm{id} \otimes \sigma_* \otimes \mathrm{id})(\mathrm{id}^2 \otimes \phi)\big[(m \otimes \circ)(\mathrm{id} \otimes \sigma_* \otimes \mathrm{id})(\mathrm{id}^2 \otimes \phi) \otimes \mathrm{id}\big]$$

$$= (m \otimes \circ)(\mathrm{id}^2 \otimes \circ \otimes \mathrm{id})(\mathrm{id} \otimes \sigma_* \otimes \mathrm{id}^2)(m \otimes \mathrm{id} \otimes \tau \otimes \mathrm{id})(\mathrm{id} \otimes \sigma_* \otimes \mathrm{id} \otimes \phi)(\mathrm{id}^2 \otimes \phi \otimes \mathrm{id})$$

$$= (m \otimes \circ)(\mathrm{id} \otimes \sigma_* \otimes m\tau^{-1}\sigma)(m \otimes \mathrm{id} \otimes \tau \otimes \mathrm{id})(\mathrm{id} \otimes \sigma_* \otimes \mathrm{id} \otimes \phi)(\mathrm{id}^2 \otimes \phi \otimes \mathrm{id})$$

$$= (m \otimes \circ)(m \otimes \sigma_* \otimes m)(\mathrm{id}^3 \otimes \sigma \otimes \mathrm{id})(\mathrm{id} \otimes \sigma_* \otimes \mathrm{id} \otimes \phi)(\mathrm{id}^2 \otimes \phi \otimes \mathrm{id})$$

$$= (m \otimes \circ)(\mathrm{id} \otimes m \otimes \mathrm{id}^2)(\mathrm{id}^2 \otimes \sigma_* \otimes m)(\mathrm{id} \otimes \sigma_* \otimes \sigma \otimes \mathrm{id})(\mathrm{id}^2 \otimes \phi \otimes \phi)$$

$$= (m \otimes \circ)(\mathrm{id} \otimes \sigma_* \otimes \mathrm{id})(\mathrm{id}^2 \otimes m \otimes m)(\mathrm{id}^3 \otimes \sigma \otimes \mathrm{id})(\mathrm{id}^2 \otimes \phi \otimes \phi)$$

$$= (m \otimes \circ)(\mathrm{id} \otimes \sigma_* \otimes \mathrm{id})(\mathrm{id}^2 \otimes \phi m) = m_\Gamma^{\mathrm{r}}(\mathrm{id} \otimes m).$$

Here, we have used (3.22)–(3.23), and identities

$$(3.29) \qquad\qquad \sigma_*(\circ \otimes \mathrm{id}) = (\mathrm{id} \otimes \circ)(\sigma_* \otimes \mathrm{id})(\mathrm{id} \otimes \tau)$$

$$(3.30) \qquad\qquad (m \otimes \mathrm{id})(\mathrm{id} \otimes \sigma_*)(\sigma_* \otimes \mathrm{id}) = \sigma_*(\mathrm{id} \otimes m),$$

which follow from (3.15) and (3.22).

The maps m_Γ^{l} and m_Γ^{r} commute, because

$$m_\Gamma^{\mathrm{r}}(m_\Gamma^{\mathrm{l}} \otimes \mathrm{id}) = m_\Gamma^{\mathrm{r}}(m \otimes \mathrm{id}^2) = (m \otimes \mathrm{id})(m \otimes \mathrm{id} \otimes \circ)(\mathrm{id}^2 \otimes \sigma_* \otimes \mathrm{id})(\mathrm{id}^3 \otimes \phi)$$

$$= (m \otimes \mathrm{id})\big[\mathrm{id} \otimes (m \otimes \circ)(\mathrm{id} \otimes \sigma_* \otimes \mathrm{id})(\mathrm{id}^2 \otimes \phi)\big] = m_\Gamma^{\mathrm{l}}(\mathrm{id} \otimes m_\Gamma^{\mathrm{r}}).$$

It is easy to see that the bimodule Γ is unital.

According to Lemma 3.9 and equation (3.22),

$$(3.31) \qquad\qquad \pi m = [\epsilon \otimes \pi + \circ(\pi \otimes \mathrm{id})]\sigma^{-1}\tau.$$

Using this, equations (3.16) and (3.18) and (3.23) we obtain

$$dm = (m \otimes \pi m)(\mathrm{id} \otimes \sigma \otimes \mathrm{id})(\phi \otimes \phi) =$$
$$= \big[m \otimes (\epsilon \otimes \pi)\sigma^{-1}\tau\big](\mathrm{id} \otimes \sigma \otimes \mathrm{id})(\phi \otimes \phi) + \big[m \otimes \circ(\pi \otimes \mathrm{id})\sigma^{-1}\tau\big](\mathrm{id} \otimes \sigma \otimes \mathrm{id})(\phi \otimes \phi)$$
$$= (m \otimes \epsilon \otimes \pi)(\mathrm{id} \otimes \tau \otimes \mathrm{id})(\phi \otimes \phi) + \big[m \otimes \circ(\pi \otimes \mathrm{id})\big](\mathrm{id} \otimes \tau \otimes \mathrm{id})(\phi \otimes \phi)$$
$$= (m \otimes \pi)(\mathrm{id} \otimes \phi) + (m \otimes \circ)(\mathrm{id} \otimes \sigma_* \otimes \mathrm{id})(\mathrm{id} \otimes \pi \otimes \mathrm{id}^2)(\phi \otimes \phi)$$
$$= m_\Gamma^{\mathrm{l}}(\mathrm{id} \otimes d) + m_\Gamma^{\mathrm{r}}(d \otimes \mathrm{id}).$$

To complete the proof, let us observe that (3.16) implies that ℓ_Γ given by (3.17) is indeed the left action. $\square$

4. Bicovariant Calculi

In this section we shall study bicovariant differential calculi Γ over G. As in the standard theory [5] the right action ϱ_Γ and the left action ℓ_Γ are mutually compatible.

Proposition 4.1. *The diagram*

(4.1)
$$
\begin{array}{ccc}
\Gamma & \xrightarrow{\ \ell_\Gamma\ } & \mathcal{A} \otimes \Gamma \\
{\scriptstyle \varrho_\Gamma}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{id} \otimes \varrho_\Gamma} \\
\Gamma \otimes \mathcal{A} & \xrightarrow[\ell_\Gamma \otimes \mathrm{id}]{} & \mathcal{A} \otimes \Gamma \otimes \mathcal{A}
\end{array}
$$

is commutative.

Proof. Applying (3.2) and (A.1) we obtain

$$(\ell_\Gamma \otimes \mathrm{id})\varrho_\Gamma \iota_\Gamma^{\mathrm{l}} = (\ell_\Gamma \otimes \mathrm{id})(\iota_\Gamma^{\mathrm{l}} \otimes m)(\mathrm{id} \otimes \sigma \otimes \mathrm{id})(\phi \otimes \phi)$$
$$= (m \otimes \iota_\Gamma^{\mathrm{l}} \otimes m)(\mathrm{id} \otimes \sigma \otimes \mathrm{id}^3)(\phi \otimes \phi \otimes \mathrm{id}^2)(\mathrm{id} \otimes \sigma \otimes \mathrm{id})(\phi \otimes \phi)$$
$$= (m \otimes \iota_\Gamma^{\mathrm{l}} \otimes m)(\mathrm{id}^3 \otimes \sigma \otimes \mathrm{id})(\mathrm{id}^2 \otimes \phi \otimes \phi)(\mathrm{id} \otimes \sigma \otimes \mathrm{id})(\phi \otimes \phi)$$
$$= (m \otimes \varrho_\Gamma \iota_\Gamma^{\mathrm{l}})(\mathrm{id} \otimes \sigma \otimes \mathrm{id})(\phi \otimes \phi) = (\mathrm{id} \otimes \varrho_\Gamma)\ell_\Gamma \iota_\Gamma^{\mathrm{l}}. \quad \square$$

As a simple consequence of (4.1) we find that the spaces Γ_{inv} and $^{inv}\Gamma$ are right/left-invariant respectively. The following proposition characterizes the corresponding restrictions of ϱ_Γ and ℓ_Γ. Let $\mathrm{ad} \colon \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ be the adjoint action of G on itself, as defined in Appendix B.

Proposition 4.2. *The following identities hold*

(4.2) $$\varrho_\Gamma \pi = (\pi \otimes \mathrm{id})\mathrm{ad}$$

(4.3) $$\ell_\Gamma \varsigma = (\mathrm{id} \otimes \varsigma)\tau(\kappa \otimes \kappa)\mathrm{ad}\kappa^{-1}.$$

Proof. We compute

$$\varrho_\Gamma \pi = \varrho_\Gamma \iota_\Gamma^l(\kappa \otimes \mathrm{id})\phi = (\iota_\Gamma^l \otimes m)(\mathrm{id} \otimes \sigma \otimes \mathrm{id})(\sigma \otimes \mathrm{id}^2)(\kappa \otimes \kappa \otimes \mathrm{id}^2)(\phi \otimes \phi)\phi$$

$$= (\iota_\Gamma^l \otimes m)(\kappa \otimes \mathrm{id} \otimes \kappa \otimes \mathrm{id})(\mathrm{id} \otimes \tau\sigma^{-1}\tau \otimes \mathrm{id})(\sigma \otimes \mathrm{id}^2)(\phi \otimes \phi)\phi$$

$$= (\iota_\Gamma^l \otimes m)(\kappa \otimes \mathrm{id} \otimes \kappa \otimes \mathrm{id})(\phi \otimes \mathrm{id}^2)(\tau \otimes \mathrm{id})(\mathrm{id} \otimes \phi)\phi$$

$$= (\pi \otimes m)(\mathrm{id} \otimes \kappa \otimes \mathrm{id})(\tau \otimes \mathrm{id})(\mathrm{id} \otimes \phi)\phi = (\pi \otimes \mathrm{id})\mathrm{ad}.$$

Completely similarly,

$$\ell_\Gamma \varsigma = \ell_\Gamma \iota_\Gamma^r(\mathrm{id} \otimes \kappa)\phi = (m \otimes \iota_\Gamma^r)(\mathrm{id} \otimes \sigma \otimes \mathrm{id})(\mathrm{id}^2 \otimes \sigma)(\mathrm{id}^2 \otimes \kappa \otimes \kappa)(\phi \otimes \phi)\phi$$

$$= (m \otimes \iota_\Gamma^r)(\mathrm{id} \otimes \kappa \otimes \mathrm{id} \otimes \kappa)(\mathrm{id} \otimes \tau\sigma^{-1}\tau \otimes \mathrm{id})(\mathrm{id}^2 \otimes \sigma)(\phi \otimes \phi)\phi$$

$$= (m \otimes \iota_\Gamma^r)(\mathrm{id} \otimes \kappa \otimes \mathrm{id} \otimes \kappa)(\mathrm{id}^2 \otimes \phi)(\mathrm{id} \otimes \tau)(\phi \otimes \mathrm{id})\phi$$

$$= (m \otimes \varsigma)(\mathrm{id} \otimes \kappa \otimes \mathrm{id})(\mathrm{id} \otimes \tau)(\phi \otimes \mathrm{id})\phi = (\mathrm{id} \otimes \varsigma)\tau(\kappa \otimes \kappa)\mathrm{ad}\kappa^{-1}. \quad \square$$

We pass to the the analysis of the specific twisting properties of the left and the right action maps.

Proposition 4.3. *The following equalities hold*

$$(4.4) \qquad\qquad (\ell_\Gamma \otimes \mathrm{id})\sigma_{n+m}^r = (\mathrm{id} \otimes \sigma_m^r)(\sigma_n \otimes \mathrm{id})(\mathrm{id} \otimes \ell_\Gamma)$$

$$(4.5) \qquad\qquad (\mathrm{id} \otimes \varrho_\Gamma)\sigma_{n+m}^l = (\sigma_m^l \otimes \mathrm{id})(\mathrm{id} \otimes \sigma_n)(\varrho_\Gamma \otimes \mathrm{id})$$

for each $n, m \in \mathbb{Z}$.

Proof. A direct computation gives

$$(\mathrm{id} \otimes \sigma_m^r)(\sigma_n \otimes \mathrm{id})(\mathrm{id} \otimes \ell_\Gamma)(\mathrm{id} \otimes \iota_\Gamma^l) =$$

$$= (\mathrm{id} \otimes \sigma_m^r)(\sigma_n \otimes \mathrm{id})(\mathrm{id} \otimes m \otimes \iota_\Gamma^l)(\mathrm{id}^2 \otimes \sigma \otimes \mathrm{id})(\mathrm{id} \otimes \phi \otimes \phi)$$

$$= (m \otimes \sigma_m^r)(\mathrm{id}^3 \otimes \iota_\Gamma^l)(\mathrm{id} \otimes \sigma_n \otimes \mathrm{id}^2)(\sigma_n \otimes \mathrm{id}^3)(\mathrm{id}^2 \otimes \sigma \otimes \mathrm{id})(\mathrm{id} \otimes \phi \otimes \phi)$$

$$= (m \otimes \iota_\Gamma^l \otimes \mathrm{id})(\mathrm{id}^3 \otimes \sigma_m)(\mathrm{id}^2 \otimes \sigma_m \otimes \mathrm{id})(\mathrm{id} \otimes \sigma_n \otimes \mathrm{id}^2)(\sigma_n \otimes \sigma \otimes \mathrm{id})(\mathrm{id} \otimes \phi \otimes \phi)$$

$$= (m \otimes \iota_\Gamma^l \otimes \mathrm{id})(\mathrm{id} \otimes \sigma \otimes \sigma_m)(\mathrm{id}^2 \otimes \sigma_n \otimes \mathrm{id})(\mathrm{id} \otimes \sigma_m \otimes \mathrm{id}^2)(\sigma_n \otimes \mathrm{id}^3)(\mathrm{id} \otimes \phi \otimes \phi)$$

$$= (m \otimes \iota_\Gamma^l \otimes \mathrm{id})(\mathrm{id} \otimes \sigma \otimes \mathrm{id}^2)(\phi \otimes \phi \otimes \mathrm{id})(\mathrm{id} \otimes \sigma_{n+m})(\sigma_{n+m} \otimes \mathrm{id})$$

$$= (\ell_\Gamma \iota_\Gamma^l \otimes \mathrm{id})(\mathrm{id} \otimes \sigma_{n+m})(\sigma_{n+m} \otimes \mathrm{id}) = (\ell_\Gamma \otimes \mathrm{id})\sigma_{n+m}^r(\mathrm{id} \otimes \iota_\Gamma^l).$$

Similarly we obtain

$$(\sigma_m^l \otimes \mathrm{id})(\mathrm{id} \otimes \sigma_n)(\varrho_\Gamma \otimes \mathrm{id})(\iota_\Gamma^r \otimes \mathrm{id}) =$$

$$= (\sigma_m^l \otimes \mathrm{id})(\mathrm{id} \otimes \sigma_n)(\iota_\Gamma^r \otimes m \otimes \mathrm{id})(\mathrm{id} \otimes \sigma \otimes \mathrm{id}^2)(\phi \otimes \phi \otimes \mathrm{id})$$

$$= (\sigma_m^l \otimes m)(\mathrm{id} \otimes \sigma_n \otimes \mathrm{id})(\iota_\Gamma^r \otimes \mathrm{id} \otimes \sigma_n)(\mathrm{id} \otimes \sigma \otimes \mathrm{id}^2)(\phi \otimes \phi \otimes \mathrm{id})$$

$$= (\mathrm{id} \otimes \iota_\Gamma^r \otimes m)(\sigma_m \otimes \mathrm{id}^3)(\mathrm{id} \otimes \sigma_m \otimes \mathrm{id}^2)(\mathrm{id}^2 \otimes \sigma_n \otimes \mathrm{id})(\mathrm{id} \otimes \sigma \otimes \sigma_n)(\phi \otimes \phi \otimes \mathrm{id})$$

$$= (\mathrm{id} \otimes \iota_\Gamma^r \otimes m)(\sigma_m \otimes \sigma \otimes \mathrm{id})(\mathrm{id} \otimes \sigma_n \otimes \mathrm{id}^2)(\mathrm{id}^2 \otimes \sigma_m \otimes \mathrm{id})(\mathrm{id}^3 \otimes \sigma_n)(\phi \otimes \phi \otimes \mathrm{id})$$

$$= (\mathrm{id} \otimes \iota_\Gamma^r \otimes m)(\sigma_m \otimes \sigma \otimes \mathrm{id})(\mathrm{id} \otimes \sigma_n \otimes \phi)(\phi \otimes \sigma_{n+m})$$

$$= (\mathrm{id} \otimes \iota_\Gamma^r \otimes m)(\mathrm{id}^2 \otimes \sigma \otimes \mathrm{id})(\mathrm{id} \otimes \phi \otimes \phi)(\sigma_{n+m} \otimes \mathrm{id})(\mathrm{id} \otimes \sigma_{n+m})$$

$$= (\mathrm{id} \otimes \varrho_\Gamma \iota_\Gamma^r)(\sigma_{n+m} \otimes \mathrm{id})(\mathrm{id} \otimes \sigma_{n+m}) = (\mathrm{id} \otimes \varrho_\Gamma)\sigma_{n+m}^l(\iota_\Gamma^r \otimes \mathrm{id}). \quad \square$$

As a simple consequence of the previous proposition we find

$$\sigma_n^{\mathrm{r}}(\mathcal{A} \otimes \Gamma_{inv}) = \Gamma_{inv} \otimes \mathcal{A} \tag{4.6}$$

$$\sigma_n^{\mathrm{l}}(^{inv}\Gamma \otimes \mathcal{A}) = \mathcal{A} \otimes {}^{inv}\Gamma. \tag{4.7}$$

The following proposition describes the corresponding restriction twistings.

Proposition 4.4. *The following identities hold*

$$\sigma_n^{\mathrm{r}}(\mathrm{id} \otimes \pi) = (\pi \otimes \mathrm{id})\tau \tag{4.8}$$

$$\sigma_n^{\mathrm{l}}(\varsigma \otimes \mathrm{id}) = (\mathrm{id} \otimes \varsigma)\tau. \tag{4.9}$$

Proof. Using standard twisting transformations we obtain

$$\begin{aligned}
\sigma_n^{\mathrm{r}}(\mathrm{id} \otimes \pi) &= (\iota_\Gamma^{\mathrm{l}} \otimes \mathrm{id})(\mathrm{id} \otimes \sigma_n)(\sigma_n \otimes \mathrm{id})(\mathrm{id} \otimes \kappa \otimes \mathrm{id})(\mathrm{id} \otimes \phi) \\
&= (\iota_\Gamma^{\mathrm{l}} \otimes \mathrm{id})(\kappa \otimes \mathrm{id}^2)(\mathrm{id} \otimes \sigma_n)(\sigma_{-n} \otimes \mathrm{id})(\mathrm{id} \otimes \phi) \\
&= (\iota_\Gamma^{\mathrm{l}} \otimes \mathrm{id})(\kappa \otimes \mathrm{id}^2)(\phi \otimes \mathrm{id})\tau = (\pi \otimes \mathrm{id})\tau.
\end{aligned}$$

The second identity can be derived in a similar manner. $\square$

Let $\mathcal{R} \subseteq \ker(\epsilon)$ be the right $\mathcal{A}_0$-ideal which canonically corresponds to Γ. In the following proposition we have characterized bicovariance in terms of $\mathcal{R}$.

Proposition 4.5. (i) *We have*

$$\mathrm{ad}(\mathcal{R}) \subseteq \mathcal{R} \otimes \mathcal{A} \tag{4.10}$$

$$\tau(\mathcal{A} \otimes \mathcal{R}) = \mathcal{R} \otimes \mathcal{A}. \tag{4.11}$$

(ii) *Conversely, if $\mathcal{R} \subseteq \ker(\epsilon)$ corresponding to a left-covariant calculus Γ is ad-invariant, then the calculus Γ is bicovariant. Moreover, in terms of the identification $\Gamma = \Gamma_{inv} \otimes \mathcal{A}$, the right action $\varrho_\Gamma \colon \Gamma \to \Gamma \otimes \mathcal{A}$ is given by*

$$\varrho_\Gamma = (\mathrm{id}^2 \otimes m)(\mathrm{id} \otimes \sigma \otimes \mathrm{id})(\varpi \otimes \phi), \tag{4.12}$$

where the map $\varpi \colon \Gamma_{inv} \to \Gamma_{inv} \otimes \mathcal{A}$ is given by

$$\varpi\pi = (\pi \otimes \mathrm{id})\mathrm{ad}. \tag{4.13}$$

Proof. The first statement of the proposition is a direct consequence of (4.2) and (4.8). Concerning the second part, it is sufficient to check that the map ξ given by the right-hand side of (4.12) satisfies (A.2)–(A.3). Using the structuralization $\Gamma = \Gamma_{inv} \otimes \mathcal{A}$ as well as equalities (3.25) and (4.13) we obtain

$$\begin{aligned}
\xi d &= -(\mathrm{id}^2 \otimes m)(\mathrm{id} \otimes \sigma \otimes \mathrm{id})(\varpi \otimes \phi)(\pi \otimes \mathrm{id})(\mathrm{id} \otimes \kappa)\phi k^{-1} \\
&= -(\pi \otimes \mathrm{id} \otimes m)(\mathrm{id} \otimes \sigma \otimes \mathrm{id})\big[\mathrm{ad} \otimes \sigma(\kappa \otimes \kappa)\phi\big]\phi\kappa^{-1} \\
&= -(\pi \otimes \sigma)(\mathrm{id} \otimes m \otimes \mathrm{id})\big(\mathrm{ad} \otimes (\kappa \otimes \kappa)\phi\big)\phi k^{-1} \\
&= -(\pi \otimes \sigma)(\mathrm{id} \otimes \kappa \otimes \kappa)(\tau \otimes \mathrm{id})(\mathrm{id} \otimes \phi)\phi k^{-1} \\
&= -(\pi \otimes \kappa \otimes \kappa)(\phi \otimes \mathrm{id})\sigma\phi k^{-1} \\
&= -(\pi \otimes \kappa \otimes \mathrm{id})(\phi\kappa^{-1} \otimes \mathrm{id})\phi = (d \otimes \mathrm{id})\phi.
\end{aligned}$$

Furthermore, (3.26) implies

$$\xi m_\Gamma^{\mathrm{r}} = (\mathrm{id}^2 \otimes m)(\mathrm{id} \otimes \sigma \otimes \mathrm{id})(\varpi \otimes \phi m) =$$
$$= (\mathrm{id}^2 \otimes m)(\mathrm{id} \otimes \sigma \otimes \mathrm{id})(\varpi \otimes m \otimes m)(\mathrm{id}^2 \otimes \sigma \otimes \mathrm{id})(\mathrm{id} \otimes \phi \otimes \phi)$$
$$= (\mathrm{id} \otimes m \otimes m)(\mathrm{id}^2 \otimes \sigma \otimes \mathrm{id})(\mathrm{id} \otimes \sigma \otimes \mathrm{id}^2)(\varpi \otimes \mathrm{id}^2 \otimes m)(\mathrm{id}^2 \otimes \sigma \otimes \mathrm{id})(\mathrm{id} \otimes \phi \otimes \phi)$$
$$= (\mathrm{id} \otimes m \otimes m)(\mathrm{id}^3 \otimes m \otimes \mathrm{id})(\mathrm{id}^2 \otimes \sigma \otimes \mathrm{id}^2)(\mathrm{id} \otimes \sigma \otimes \sigma \otimes \mathrm{id})(\varpi \otimes \phi \otimes \phi)$$
$$= (\mathrm{id} \otimes m \otimes m)(\mathrm{id}^2 \otimes \sigma \otimes \mathrm{id})(\mathrm{id}^2 \otimes m \otimes \mathrm{id}^2)(\mathrm{id} \otimes \sigma \otimes \mathrm{id}^3)(\varpi \otimes \phi \otimes \phi)$$
$$= (m_\Gamma^{\mathrm{r}} \otimes m)(\mathrm{id} \otimes \sigma \otimes \mathrm{id})(\xi \otimes \phi).$$

Consequently, Γ is bicovariant and $\xi = \varrho_\Gamma$. $\quad\square$

5. ANTIPODALLY COVARIANT CALCULI

In this Section we shall consider differential structures covariant relative to the antipode map.

Definition 7. A first-order calculus Γ is called κ-covariant iff the following equivalence holds
$$\omega \in \ker(\iota_\Gamma^{\mathrm{l}}) \iff \omega \in \ker\big[\iota_\Gamma^{\mathrm{r}}(\kappa \otimes \kappa)\tau\sigma^{-1}\tau\sigma^{-1}\tau\big].$$

Let us assume that Γ is κ-covariant. Then the formula

$$(5.1) \qquad\qquad \varkappa\iota_\Gamma^{\mathrm{l}} = \iota_\Gamma^{\mathrm{r}}(\kappa \otimes \kappa)\tau\sigma^{-1}\tau\sigma^{-1}\tau$$

consistently and uniquely determines a bijective map $\varkappa\colon \Gamma \to \Gamma$. It follows that

$$(5.2) \qquad\qquad d\kappa = \varkappa d$$

$$(5.3) \qquad\qquad \varkappa\iota_\Gamma^{\mathrm{r}} = \iota_\Gamma^{\mathrm{l}}(\kappa \otimes \kappa)\tau\sigma^{-1}\tau\sigma^{-1}\tau.$$

Let us analyze properties of Γ, in the case when it is also σ-covariant.

Proposition 5.1. *(i) If Γ is left σ-covariant (and accordingly, left $\mathcal{F}$-covariant) then*

$$(5.4) \qquad\qquad \sigma_n^{\mathrm{l}}(\varkappa \otimes \mathrm{id}) = (\mathrm{id} \otimes \varkappa)\sigma_{-n}^{\mathrm{l}}$$

$$(5.5) \qquad\qquad \varkappa m_\Gamma^{\mathrm{r}} = m_\Gamma^{\mathrm{l}}(\kappa \otimes \varkappa)(\tau\sigma^{-1}\tau\sigma^{-1}\tau)^{\mathrm{l}}.$$

(ii) If Γ is right $\mathcal{F}$-covariant then

$$(5.6) \qquad\qquad \sigma_n^{\mathrm{r}}(\mathrm{id} \otimes \varkappa) = (\varkappa \otimes \mathrm{id})\sigma_{-n}^{\mathrm{r}}$$

$$(5.7) \qquad\qquad \varkappa m_\Gamma^{\mathrm{l}} = m_\Gamma^{\mathrm{r}}(\varkappa \otimes \kappa)(\tau\sigma^{-1}\tau\sigma^{-1}\tau)^{\mathrm{r}}.$$

Proof. Let us assume that Γ is left $\mathcal{F}$-covariant. A direct computation gives

$$\sigma_n^{\mathrm{l}}(\varkappa \otimes \mathrm{id})(\iota_\Gamma^{\mathrm{l}} \otimes \mathrm{id}) = \sigma_n^{\mathrm{l}}(\iota_\Gamma^{\mathrm{r}} \otimes \mathrm{id})(\kappa \otimes \kappa \otimes \mathrm{id})(\tau\sigma^{-1}\tau\sigma^{-1}\tau \otimes \mathrm{id})$$
$$= (\mathrm{id} \otimes \iota_\Gamma^{\mathrm{r}})(\sigma_n \otimes \mathrm{id})(\mathrm{id} \otimes \sigma_n)(\kappa \otimes \kappa \otimes \mathrm{id})(\tau\sigma^{-1}\tau\sigma^{-1}\tau \otimes \mathrm{id})$$
$$= (\mathrm{id} \otimes \iota_\Gamma^{\mathrm{r}})(\mathrm{id} \otimes \kappa \otimes \kappa)(\mathrm{id} \otimes \tau\sigma^{-1}\tau\sigma^{-1}\tau)(\sigma_{-n} \otimes \mathrm{id})(\mathrm{id} \otimes \sigma_{-n})$$
$$= (\mathrm{id} \otimes \varkappa\iota_\Gamma^{\mathrm{l}})(\sigma_{-n} \otimes \mathrm{id})(\mathrm{id} \otimes \sigma_{-n}) = (\mathrm{id} \otimes \varkappa)\sigma_{-n}^{\mathrm{l}}(\iota_\Gamma^{\mathrm{l}} \otimes \mathrm{id}).$$

Furthermore,

$$\varkappa m_\Gamma^r(\iota_\Gamma^r \otimes \mathrm{id}) = \iota_\Gamma^l(\kappa \otimes \kappa)\sigma_{-2}(\mathrm{id} \otimes m) = \iota_\Gamma^l(\kappa m \otimes \kappa)(\mathrm{id} \otimes \sigma_{-2})(\sigma_{-2} \otimes \mathrm{id})$$
$$= \iota_\Gamma^l(m \otimes \mathrm{id})(\kappa \otimes \kappa \otimes \kappa)(\sigma_{-2} \otimes \mathrm{id})(\mathrm{id} \otimes \sigma_{-2})(\sigma_{-2} \otimes \mathrm{id})$$
$$= m_\Gamma^l(\mathrm{id} \otimes \iota_\Gamma^l)(\kappa \otimes \kappa \otimes \kappa)(\mathrm{id} \otimes \sigma_{-2})(\sigma_{-2} \otimes \mathrm{id})(\sigma_{-2} \otimes \mathrm{id})$$
$$= m_\Gamma^l(\kappa \otimes \varkappa\iota_\Gamma^r)(\sigma_{-2} \otimes \mathrm{id})(\sigma_{-2} \otimes \mathrm{id}) = m_\Gamma^l(\kappa \otimes \varkappa)\sigma_{-2}^l(\iota_\Gamma^l \otimes \mathrm{id}).$$

Symmetrically, assuming the right $\mathcal{F}$-covariance of Γ we get

$$\sigma_n^r(\mathrm{id} \otimes \varkappa)(\mathrm{id} \otimes \iota_\Gamma^r) = \sigma_n^r(\mathrm{id} \otimes \iota_\Gamma^l)(\mathrm{id} \otimes \kappa \otimes \kappa)(\mathrm{id} \otimes \sigma_{-2})$$
$$= (\iota_\Gamma^l \otimes \mathrm{id})(\mathrm{id} \otimes \sigma_n)(\sigma_n \otimes \mathrm{id})(\mathrm{id} \otimes \kappa \otimes \kappa)(\mathrm{id} \otimes \sigma_{-2})$$
$$= (\iota_\Gamma^l \otimes \mathrm{id})(\kappa \otimes \kappa \otimes \mathrm{id})(\sigma_{-2} \otimes \mathrm{id})(\mathrm{id} \otimes \sigma_{-n})(\sigma_{-n} \otimes \mathrm{id})$$
$$= (\varkappa \otimes \mathrm{id})(\iota_\Gamma^r \otimes \mathrm{id})(\mathrm{id} \otimes \sigma_{-n})(\sigma_{-n} \otimes \mathrm{id})$$
$$= (\varkappa \otimes \mathrm{id})\sigma_{-n}^r(\mathrm{id} \otimes \iota_\Gamma^r).$$

Finally,

$$\varkappa m_\Gamma^l(\mathrm{id} \otimes \iota_\Gamma^l) = \iota_\Gamma^r(\kappa \otimes \kappa)\sigma_{-2}(m \otimes \mathrm{id}) = \iota_\Gamma^r(\kappa \otimes \kappa m)(\sigma_{-2} \otimes \mathrm{id})(\mathrm{id} \otimes \sigma_{-2})$$
$$= \iota_\Gamma^r(\mathrm{id} \otimes m)(\kappa \otimes \kappa \otimes \kappa)(\mathrm{id} \otimes \sigma_{-2})(\sigma_{-2} \otimes \mathrm{id})(\mathrm{id} \otimes \sigma_{-2})$$
$$= m_\Gamma^r(\iota_\Gamma^r \otimes \mathrm{id})(\kappa \otimes \kappa \otimes \kappa)(\sigma_{-2} \otimes \mathrm{id})(\mathrm{id} \otimes \sigma_{-2})(\sigma_{-2} \otimes \mathrm{id})$$
$$= m_\Gamma^r(\varkappa\iota_\Gamma^l \otimes \kappa)(\mathrm{id} \otimes \sigma_{-2})(\sigma_{-2} \otimes \mathrm{id}) = m_\Gamma^r(\varkappa \otimes \kappa)\sigma_{-2}^r(\mathrm{id} \otimes \iota_\Gamma^l),$$

which completes the proof. $\square$

Now, we shall analyze interrelations between κ-covariance and bicovariance.

Proposition 5.2. *A left-covariant calculus Γ is κ-covariant if and only if it is bicovariant. In this case the following identities hold:*

(5.8) $$\ell_\Gamma \varkappa = (\kappa \otimes \varkappa)\sigma^l \varrho_\Gamma$$
(5.9) $$\varrho_\Gamma \varkappa = (\varkappa \otimes \kappa)\sigma^r \ell_\Gamma.$$

Moreover, the diagram

(5.10)
$$\begin{array}{ccc} \mathcal{A} \otimes \Gamma \otimes \mathcal{A} & \xrightarrow{\kappa \otimes \mathrm{id} \otimes \kappa} & \mathcal{A} \otimes \Gamma \otimes \mathcal{A} \\ \uparrow & & \downarrow \\ \Gamma & \xrightarrow{-\varkappa} & \Gamma \end{array}$$

is commutative. Here, the vertical arrows are the corresponding double-sided actions and products.

Proof. Let us assume that Γ is left-covariant and κ-covariant, and let us consider a map $\xi \colon \Gamma \to \Gamma \otimes \mathcal{A}$ defined by

$$\xi = (\varkappa^{-1} \otimes \kappa^{-1})(\sigma^l)^{-1}\ell_\Gamma \varkappa.$$

It turns out that ξ is the right action for Γ. Indeed,

$$\xi \iota_\Gamma^{\mathrm{r}} = (\varkappa^{-1} \otimes \kappa^{-1})(\sigma^{\mathrm{l}})^{-1} \ell_\Gamma \iota_\Gamma^{\mathrm{l}} (\kappa \otimes \kappa) \sigma_{-2} =$$

$$= (\varkappa^{-1} \otimes \kappa^{-1})(\sigma^{\mathrm{l}})^{-1}(m \otimes \iota_\Gamma^{\mathrm{l}})(\mathrm{id} \otimes \sigma \otimes \mathrm{id})(\phi\kappa \otimes \phi\kappa)\sigma_{-2}$$

$$= (\varkappa^{-1} \otimes \kappa^{-1})(\iota_\Gamma^{\mathrm{l}} \otimes \mathrm{id})(\mathrm{id} \otimes \sigma^{-1})(\sigma^{-1} \otimes \mathrm{id})(m \otimes \mathrm{id}^2)(\mathrm{id} \otimes \sigma \otimes \mathrm{id})(\phi\kappa \otimes \phi\kappa)\sigma_{-2}$$

$$= (\varkappa^{-1} \otimes \kappa^{-1})(\iota_\Gamma^{\mathrm{l}} \otimes \mathrm{id})(\mathrm{id} \otimes \sigma^{-1})(\mathrm{id} \otimes m \otimes \mathrm{id})(\sigma^{-1} \otimes \mathrm{id}^2)(\phi\kappa \otimes \phi\kappa)\sigma_{-2}$$

$$= (\varkappa^{-1} \otimes \kappa^{-1})(\iota_\Gamma^{\mathrm{l}} \otimes m)(\mathrm{id} \otimes \sigma^{-1} \otimes \mathrm{id})(\kappa \otimes \kappa \otimes \kappa \otimes \kappa)(\phi \otimes \phi)\sigma_{-2}$$

$$= (\iota_\Gamma^{\mathrm{r}} \otimes m)[(\sigma_{-2})^{-1} \otimes (\sigma_{-2})^{-1}](\mathrm{id} \otimes \sigma^{-1} \otimes \mathrm{id})(\phi \otimes \phi)\sigma_{-2}$$

$$= (\iota_\Gamma^{\mathrm{r}} \otimes m)[(\sigma_{-2})^{-1} \otimes \mathrm{id}^2](\mathrm{id} \otimes \phi \otimes \mathrm{id})[\mathrm{id} \otimes (\sigma_{-1})^{-1}](\phi \otimes \mathrm{id})\sigma_{-2}$$

$$= (\iota_\Gamma^{\mathrm{r}} \otimes m)[(\sigma_{-2})^{-1} \otimes \mathrm{id}](\mathrm{id} \otimes \phi \otimes \mathrm{id})(\sigma_{-1} \otimes \mathrm{id})(\mathrm{id} \otimes \phi) = (\iota_\Gamma^{\mathrm{r}} \otimes m)(\mathrm{id} \otimes \sigma \otimes \mathrm{id})(\phi \otimes \phi).$$

Consequently Γ is right-covariant with $\xi = \varrho_\Gamma$ and (5.8) holds.

Similarly, if Γ is κ-covariant and right-covariant then a map $\xi \colon \Gamma \to \mathcal{A} \otimes \Gamma$ given by

$$\xi = (\kappa^{-1} \otimes \varkappa^{-1})(\sigma^{\mathrm{r}})^{-1} \varrho_\Gamma \varkappa$$

satisfies

$$\xi \iota_\Gamma^{\mathrm{l}} = (\kappa^{-1} \otimes \varkappa^{-1})(\sigma^{\mathrm{r}})^{-1} \varrho_\Gamma \iota_\Gamma^{\mathrm{r}} (\kappa \otimes \kappa) \sigma_{-2}$$

$$= (\kappa^{-1} \otimes \varkappa^{-1})(\sigma^{\mathrm{r}})^{-1}(\iota_\Gamma^{\mathrm{r}} \otimes m)(\mathrm{id} \otimes \sigma \otimes \mathrm{id})(\phi\kappa \otimes \phi\kappa)\sigma_{-2}$$

$$= (\kappa^{-1} \otimes \varkappa^{-1})(m \otimes \iota_\Gamma^{\mathrm{r}})(\mathrm{id} \otimes \sigma^{-1} \otimes \mathrm{id})(\kappa \otimes \kappa \otimes \kappa \otimes \kappa)(\phi \otimes \phi)\sigma_{-2}$$

$$= (m \otimes \iota_\Gamma^{\mathrm{l}})[(\sigma_{-2})^{-1} \otimes (\sigma_{-2})^{-1}](\mathrm{id} \otimes \sigma^{-1} \otimes \mathrm{id})(\phi \otimes \phi)\sigma_{-2}$$

$$= (m \otimes \iota_\Gamma^{\mathrm{l}})(\mathrm{id} \otimes \sigma \otimes \mathrm{id})(\phi \otimes \phi).$$

This implies that Γ is also left-covariant with $\xi = \ell_\Gamma$, and equality (5.9) holds.

Finally, let us assume that Γ is bicovariant and consider a map $\varkappa \colon \Gamma \to \Gamma$ defined by diagram (5.10). Then a straightforward computation shows that equality (5.1) holds, and that $\varkappa$ is bijective. In other words, Γ is κ-covariant and (5.10) holds by construction. $\square$

Let us assume that Γ is bicovariant. It turns out that quadruplets $(\Gamma_{inv}, \pi, \circ, \mathcal{R})$ and $(^{inv}\Gamma, \varsigma, \bullet, \mathcal{K})$ corresponding to the left/right-covariant structure on Γ are naturally related via the antipodal maps. According to (5.8)–(5.9)

$$(5.11) \qquad\qquad \varkappa(\Gamma_{inv}) = {}^{inv}\Gamma \qquad \varkappa({}^{inv}\Gamma) = \Gamma_{inv}.$$

Proposition 5.3. *The following identities hold*

$$(5.12) \qquad\qquad \varkappa\pi = \varsigma\kappa_0 \qquad \varkappa\varsigma = \pi\kappa_0$$

$$(5.13) \qquad\qquad \varkappa\circ(\pi \otimes \mathrm{id}) = \bullet(\kappa_0 \otimes \varsigma\kappa_0)\tau$$

$$(5.14) \qquad\qquad \varkappa\bullet(\mathrm{id} \otimes \varsigma) = \circ(\pi\kappa_0 \otimes \kappa_0)\tau$$

$$(5.15) \qquad\qquad \kappa_0(\mathcal{R}) = \mathcal{K} \qquad \kappa_0(\mathcal{K}) = \mathcal{R}$$

where $\kappa_0 = (\epsilon \otimes \kappa)\sigma\phi = (\kappa \otimes \epsilon)\sigma\phi$ is the antipode associated to $\mathcal{A}_0$.

Proof. A direct computation gives

$$\varkappa\pi = \varkappa\iota_\Gamma^l(\kappa\otimes\mathrm{id})\phi = \iota_\Gamma^r\sigma_{-2}(\kappa^2\otimes\kappa)\phi = \iota_\Gamma^r(\mathrm{id}\otimes\kappa)\sigma_2(\kappa\otimes\kappa)\phi = m_\Gamma^r(d\otimes\kappa)\sigma\tau^{-1}\phi\kappa$$

$$= m_\Gamma^r\big[m_\Gamma^r(\varsigma\otimes\mathrm{id})\phi\otimes\kappa\big]\sigma\tau^{-1}\phi\kappa = m_\Gamma^r(\mathrm{id}\otimes m)(\varsigma\otimes\mathrm{id}\otimes\kappa)(\mathrm{id}\otimes\sigma\tau^{-1})(\phi\otimes\mathrm{id})\phi$$

$$= m_\Gamma^r(\mathrm{id}\otimes m)(\varsigma\otimes\mathrm{id}\otimes\kappa)(\mathrm{id}\otimes\sigma\tau^{-1})(\mathrm{id}\otimes\phi)\phi = m_\Gamma^r(\varsigma\otimes 1\epsilon\kappa^{-1})\phi\kappa = \varsigma(\kappa\otimes\epsilon)\sigma\phi.$$

Similarly,

$$\varkappa\varsigma = \varkappa\iota_\Gamma^r(\mathrm{id}\otimes\kappa)\phi = \iota_\Gamma^l\sigma_{-2}(\kappa\otimes\kappa^2)\phi = \iota_\Gamma^l(\kappa\otimes\mathrm{id})\sigma_2(\kappa\otimes\kappa)\sigma = m_\Gamma^l(\kappa\otimes d)\sigma\tau^{-1}\phi\kappa$$

$$= m_\Gamma^l\big[\kappa\otimes m_\Gamma^l(\mathrm{id}\otimes\pi)\phi\big]\sigma\tau^{-1}\phi\kappa = m_\Gamma^l(m\otimes\pi)(\kappa\otimes\mathrm{id}^2)(\sigma\tau^{-1}\otimes\mathrm{id})(\phi\otimes\mathrm{id})\phi$$

$$= m_\Gamma^l(1\epsilon\kappa^{-1}\otimes\pi)\phi = \pi(\epsilon\otimes\kappa)\sigma\phi = \pi\kappa_0.$$

Relations (5.15) immediately follow from (5.12), definition of spaces $\mathcal{R}$ and $\mathcal{K}$ and the fact that $\epsilon\kappa_0 = \epsilon$.

Let us check (5.13)–(5.14). On the space $\ker(\epsilon)\otimes\mathcal{A}$ the following equalities hold

$$\varkappa\circ(\pi\otimes\mathrm{id}) = \varkappa\pi m\tau^{-1}\sigma = \varsigma\kappa_0 m_0 = \varsigma m_0(\kappa_0\otimes\kappa_0)\tau = \bullet(\kappa_0\otimes\varsigma\kappa_0)\tau.$$

Similarly, in the framework of the space $\mathcal{A}\otimes\ker(\epsilon)$ we can write

$$\varkappa\bullet(\mathrm{id}\otimes\varsigma) = \varkappa\varsigma m_0 = \pi\kappa_0 m_0 = \pi m_0(\kappa_0\otimes\kappa_0)\tau = \circ(\pi\kappa_0\otimes\kappa_0)\tau. \quad \square$$

In terms of the bimodule structuralizations $\Gamma\leftrightarrow\mathcal{A}\otimes\Gamma_{inv}$ and $^{inv}\Gamma\otimes\mathcal{A}\leftrightarrow\Gamma$ the operator $\varkappa$ has a particularly simple form.

Proposition 5.4. *The following identities hold*

$$(5.16) \qquad\qquad \varkappa m_\Gamma^l(\mathrm{id}\otimes\pi) = m_\Gamma^r(\varsigma\kappa_0\otimes\kappa)\tau$$

$$(5.17) \qquad\qquad \varkappa m_\Gamma^r(\varsigma\otimes\mathrm{id}) = m_\Gamma^l(\kappa\otimes\pi\kappa_0)\tau.$$

Proof. We compute

$$\varkappa m_\Gamma^l(\mathrm{id}\otimes\pi) = m_\Gamma^r(\sigma_{-2}^r)(\kappa\otimes\varkappa\pi) = m_\Gamma^r(\sigma_{-2}^r)(\kappa\otimes\varsigma\kappa_0)$$

$$= m_\Gamma^r(\varsigma\otimes\mathrm{id})\tau(\kappa\otimes\kappa_0) = m_\Gamma^r(\varsigma\kappa_0\otimes\kappa)\tau.$$

Similarly,

$$\varkappa m_\Gamma^r(\varsigma\otimes\mathrm{id}) = (m_\Gamma^l)\sigma_{-2}^l(\varkappa\varsigma\otimes\kappa) = (m_\Gamma^l)\sigma_{-2}^l(\pi\kappa_0\otimes\kappa)$$

$$= m_\Gamma^l(\mathrm{id}\otimes\pi)\tau(\kappa_0\otimes\kappa) = m_\Gamma^l(\kappa\otimes\pi\kappa_0)\tau. \quad \square$$

6. On $*$-Covariant Differential Structures

Let us consider a quantum space X, represented by a unital algebra $\mathcal{A}$ and assume that X is $\mathcal{T}$-braided. Let us also assume that $\mathcal{A}$ is equipped with a $*$-structure such that

$$(6.1) \qquad\qquad (*\otimes *)\alpha = \psi\alpha^{-1}\psi(*\otimes *),$$

for each $\alpha\in\mathcal{T}$. Here, $\psi\colon\mathcal{A}\otimes\mathcal{A}\to\mathcal{A}\otimes\mathcal{A}$ is the standard transposition. It is easy to see that then (6.1) holds for every $\alpha\in\mathcal{T}^*$. It is worth noticing that the operators

$$\mathcal{T}_c = \Big\{\psi\alpha^{-1}\psi \mid \alpha\in\mathcal{T}\Big\}$$

also form a braid system over $\mathcal{A}$.

Proposition 6.1. *Let us consider a *-covariant calculus Γ over X. Then*

(i) *If Γ is left $\mathcal{T}$-covariant then it is also left $\mathcal{T}_c$-covariant and*

$$\alpha^{\mathrm{l}}(* \otimes *) = (* \otimes *)(\psi \alpha^{-1} \psi)^{\mathrm{l}}, \tag{6.2}$$

for each $\alpha \in \mathcal{T}$.

(ii) *Similarly, if Γ is right $\mathcal{T}$-covariant then it is right $\mathcal{T}_c$-covariant, with*

$$\alpha^{\mathrm{r}}(* \otimes *) = (* \otimes *)(\psi \alpha^{-1} \psi)^{\mathrm{r}}, \tag{6.3}$$

for each $\alpha \in \mathcal{T}$. $\square$

Let us now switch to multi-braided quantum groups G. Let us assume that the *-structure on $\mathcal{A}$ satisfies

$$\phi* = (* \otimes *)\psi\sigma^{-1}\phi. \tag{6.4}$$

Definition 8. We shall say that the antimultiplicative *-involution on $\mathcal{A}$ satisfying the above equality is a *-structure on a braided quantum group G.

This implies a number of further compatibility relations between $*$ and maps appearing at the group level. At first, we have

$$\phi*\kappa = \big((*\kappa) \otimes (*\kappa)\big)\psi\phi. \tag{6.5}$$

The above equality implies

$$\epsilon(a)^* = \epsilon(\kappa(a)^*). \tag{6.6}$$

Furthermore, as in the classical theory we have

$$\kappa^{-1}(a) = \kappa(a^*)^* \tag{6.7}$$

for each $a \in \mathcal{A}$. Indeed,

$$\epsilon(a)1 = m\psi(*\kappa*\kappa \otimes \kappa)\psi\phi(a) = m\psi(\kappa \otimes *\kappa*\kappa)\psi\phi(a) = \kappa[\kappa(a^{(1)})^*]^*\kappa(a^{(2)})$$
$$= \kappa(a^{(1)})\kappa[\kappa(a^{(2)})^*]^*,$$

and consequently

$$a = \epsilon(a^{(1)})a^{(2)} = \kappa[\kappa(a^{(1)})^*]^*\kappa(a^{(2)})a^{(3)} = \kappa[\kappa(a^{(1)})^*]^*\epsilon(a^{(2)}) = \kappa[\kappa(a)^*]^*.$$

Furthermore, let us examine interrelations between $*$, and braid operators τ and σ.

Proposition 6.2. *The following identities hold*

$$\sigma(* \otimes *) = (* \otimes *)\psi\sigma^{-1}\psi \tag{6.8}$$
$$\tau(* \otimes *) = (* \otimes *)\psi\tau^{-1}\psi. \tag{6.9}$$

Proof. Direct transformations give

$$\sigma(*\kappa \otimes *\kappa) = (* \otimes *)(m\psi \otimes m\psi)\big(\mathrm{id} \otimes \psi\sigma^{-1}\phi m(\kappa \otimes \kappa)\psi \otimes \mathrm{id}\big)(\psi\phi \otimes \psi\phi)$$

$$= (* \otimes *)\psi(m \otimes m)(\mathrm{id} \otimes \sigma^{-1}\phi m(\kappa \otimes \kappa) \otimes \mathrm{id})F$$

$$= (* \otimes *)\psi(m \otimes m)\big[\mathrm{id} \otimes \sigma^{-1}(m \otimes m)(\mathrm{id} \otimes \sigma \otimes \mathrm{id})(\phi\kappa \otimes \phi\kappa) \otimes \mathrm{id}\big]F$$

$$= (* \otimes *)\psi(m \otimes m)(\mathrm{id} \otimes m \otimes m \otimes \mathrm{id})\big[\mathrm{id} \otimes (\mathrm{id} \otimes \sigma^{-1} \otimes \mathrm{id})(\sigma^{-1} \otimes \sigma^{-1})(\phi\kappa \otimes \phi\kappa) \otimes \mathrm{id}\big]F$$

$$= (* \otimes *)\psi(m \otimes m)(m \otimes \sigma^{-1} \otimes m)(\mathrm{id} \otimes \kappa \otimes \kappa \otimes \kappa \otimes \kappa \otimes \mathrm{id})(\phi \otimes \mathrm{id}^2 \otimes \phi)F$$

$$= (* \otimes *)\psi(m \otimes m)(\mathrm{id} \otimes \sigma^{-1} \otimes \mathrm{id})(1\epsilon \otimes \kappa \otimes \kappa \otimes 1\epsilon)(\phi \otimes \phi)\psi = (* \otimes *)\psi\sigma^{-1}\psi(\kappa \otimes \kappa),$$

where $F = (\phi \otimes \phi)\psi$. This proves (6.8). Furthermore, applying (6.8) and the definition of τ we obtain

$$\tau(* \otimes *) = (*\kappa \otimes *\kappa \otimes *\epsilon)(\mathrm{id} \otimes \psi\sigma\psi)(\psi\phi \otimes \mathrm{id})\psi\sigma^{-1}\psi(\kappa^{-1} \otimes \kappa^{-1})$$

$$= (*\kappa \otimes *\kappa \otimes *\epsilon)(\mathrm{id} \otimes \psi\sigma\psi)(\psi \otimes \mathrm{id})(\mathrm{id} \otimes \psi)(\psi \otimes \mathrm{id})(\mathrm{id} \otimes \phi)\sigma^{-1}\psi(\kappa^{-1} \otimes \kappa^{-1})$$

$$= (*\kappa \otimes *\kappa \otimes *\epsilon)(\mathrm{id} \otimes \psi\sigma)(\psi \otimes \mathrm{id})(\mathrm{id} \otimes \psi\phi)\sigma^{-1}\psi(\kappa^{-1} \otimes \kappa^{-1})$$

$$= (*\kappa \otimes *\kappa \otimes *\epsilon)(\mathrm{id} \otimes \psi)(\psi \otimes \mathrm{id})(\mathrm{id} \otimes \psi)(\sigma \otimes \mathrm{id})(\mathrm{id} \otimes \phi)\sigma^{-1}\psi(\kappa^{-1} \otimes \kappa^{-1})$$

$$= (*\kappa \otimes *\kappa)(\epsilon \otimes \psi\tau^{-1})(\phi \otimes \mathrm{id})\psi(\kappa^{-1} \otimes \kappa^{-1}) = (* \otimes *)\psi\tau^{-1}\psi.$$

This completes the proof. $\square$

Condition (6.4) says that the comultiplication ϕ is a hermitian map, if $\mathcal{A} \otimes \mathcal{A}$ is endowed with the *-structure induced by σ and $*\colon \mathcal{A} \to \mathcal{A}$.

Proposition 6.3. *Let us consider a *-covariant differential calculus Γ over G.*

(i) Let us assume that Γ is, in addition, left-covariant. Then

$$(6.10) \qquad \ell_\Gamma* = \sigma^{\mathrm{l}}\psi^{\mathrm{r}}(* \otimes *)\ell_\Gamma$$

$$(6.11) \qquad *\pi = -\pi*\kappa.$$

(ii) Similarly, if Γ is right-covariant then

$$(6.12) \qquad \varrho_\Gamma* = \sigma^{\mathrm{r}}\psi^{\mathrm{l}}(* \otimes *)\varrho_\Gamma$$

$$(6.13) \qquad *\varsigma = -\varsigma*\kappa.$$

(iii) If Γ is κ-covariant then

$$(6.14) \qquad \varkappa(\vartheta^*) = \varkappa^{-1}(\vartheta)^*$$

for each $\vartheta \in \Gamma$. $\square$

For the end of this section, let us characterize *-covariance of a left-covariant calculus in terms of the corresponding right $\mathcal{A}_0$-ideal. It turns out that the characterization is the same as in the standard theory.

Proposition 6.4. *Let Γ be an arbitrary left-covariant calculus over G and $\mathcal{R} \subseteq \ker(\epsilon)$ the associated right $\mathcal{A}_0$-ideal. Then the calculus Γ is *-covariant if and only if $\mathcal{R}$ is $*\kappa$-invariant.*

Proof. If Γ is *-covariant then (6.11), together with the definition of $\mathcal{R}$, implies that $\mathcal{R}$ is $*\kappa$-invariant.

Conversely, let us assume that $\mathcal{R}$ is $*\kappa$-invariant. Then the formula (6.11) consistently defines an antilinear involution $*\colon \Gamma_{inv} \to \Gamma_{inv}$

According to Proposition 3.8 the formula $(a\vartheta)^* = \vartheta^* a^*$, where $a \in \mathcal{A}$ and $\vartheta \in \Gamma_{inv}$ consistently defines an antilinear extension $*\colon \Gamma \to \Gamma$. Applying the elementary transformations with d and π we obtain

$$
\begin{aligned}
(da)^* = [a^{(1)}\pi(a^{(2)})]^* &= -\pi(\kappa(a^{(2)})^*)a^{(1)*} \\
&= -\pi(\kappa(a^{(2)})^*)\kappa(\kappa(a^{(1)})^*) \\
&= -\kappa(\kappa(a^{(3)})^*)d[\kappa(a^{(2)})^*]\kappa(\kappa(a^{(1)})^*) \\
&= a^{(3)*}\kappa(a^{(2)})^*d(a^{(1)*}) = \epsilon(a^{(2)})^*d(a^{(1)*}) = d(a^*).
\end{aligned}
$$

Consequently, Γ is *-covariant. $\square$

Let us observe that the above proof is the same as in the standard theory [5] (braidings are not included). A similar characterization of *-covariance holds for right-covariant structures.

Appendix A. Right-Covariant Calculi

Let Γ be a right-covariant first order differential calculus over a braided quantum group G. The corresponding right action $\varrho_\Gamma\colon \Gamma \to \Gamma \otimes \mathcal{A}$ can be also characterized by

$$
\text{(A.1)} \qquad \varrho_\Gamma \iota_\Gamma^{\mathrm{l}} = (\iota_\Gamma^{\mathrm{l}} \otimes m)(\mathrm{id} \otimes \sigma \otimes \mathrm{id})(\phi \otimes \phi).
$$

The right action map satisfies equalities

$$
\text{(A.2)} \qquad \varrho_\Gamma m_\Gamma^{\mathrm{r}} = (m_\Gamma^{\mathrm{r}} \otimes m)(\mathrm{id} \otimes \sigma \otimes \mathrm{id})(\varrho_\Gamma \otimes \phi)
$$
$$
\text{(A.3)} \qquad \varrho_\Gamma d = (d \otimes \mathrm{id})\phi
$$
$$
\text{(A.4)} \qquad (\mathrm{id} \otimes \epsilon)\varrho_\Gamma = \mathrm{id}
$$
$$
\text{(A.5)} \qquad (\varrho_\Gamma \otimes \mathrm{id})\varrho_\Gamma = (\mathrm{id} \otimes \phi)\varrho_\Gamma.
$$

Every right-covariant calculus is automatically right $\mathcal{F}$-covariant. In particular, the flip-over operator $\sigma^{\mathrm{r}}\colon \mathcal{A} \otimes \Gamma \to \Gamma \otimes \mathcal{A}$ is determined by the diagram

$$
\text{(A.6)} \qquad
\begin{array}{ccc}
\mathcal{A} \otimes \mathcal{A} \otimes \Gamma \otimes \mathcal{A} & \xrightarrow{\kappa \otimes \varrho_\Gamma m_\Gamma^{\mathrm{l}} \otimes \kappa} & \mathcal{A} \otimes \Gamma \otimes \mathcal{A} \otimes \mathcal{A} \\
\phi \otimes \varrho_\Gamma \uparrow & & \downarrow m_\Gamma^{\mathrm{l}} \otimes m \\
\mathcal{A} \otimes \Gamma & \xrightarrow[\sigma^{\mathrm{r}}]{} & \Gamma \otimes \mathcal{A}
\end{array}
$$

The operator σ^{r} expresses the left multiplicativity of ϱ_Γ, via the diagram

$$
\text{(A.7)} \qquad
\begin{array}{ccc}
\mathcal{A} \otimes \mathcal{A} \otimes \Gamma \otimes \mathcal{A} & \xrightarrow{\mathrm{id} \otimes \sigma^{\mathrm{r}} \otimes \mathrm{id}} & \mathcal{A} \otimes \Gamma \otimes \mathcal{A} \otimes \mathcal{A} \\
\phi \otimes \varrho_\Gamma \uparrow & & \downarrow m_\Gamma^{\mathrm{l}} \otimes m \\
\mathcal{A} \otimes \Gamma & \xrightarrow[\varrho_\Gamma m_\Gamma^{\mathrm{l}}]{} & \Gamma \otimes \mathcal{A}
\end{array}
$$

The following twisting properties hold:

$$(\varrho_\Gamma \otimes \mathrm{id})\sigma^{\mathrm r}_{n+m} = (\mathrm{id} \otimes \sigma_n)(\sigma^{\mathrm r}_m \otimes \mathrm{id})(\mathrm{id} \otimes \varrho_\Gamma). \tag{A.8}$$

Let $^{inv}\Gamma$ be the set of all right-invariant elements of Γ. Then the map $Q\colon \Gamma \to \Gamma$ defined by

$$Q = m^{\mathrm r}_\Gamma(\mathrm{id} \otimes \kappa)\varrho_\Gamma \tag{A.9}$$

projects Γ onto $^{inv}\Gamma$. Moreover,

$$Q\iota^{\mathrm r}_\Gamma = (Qd \otimes \epsilon)\sigma^{-1}\tau. \tag{A.10}$$

The composition

$$\varsigma = Qd = m^{\mathrm r}_\Gamma(d \otimes \kappa)\phi\colon \mathcal{A} \to {}^{inv}\Gamma \tag{A.11}$$

is surjective. All flip-over operators $\sigma^{\mathrm r}_n$ map $\mathcal{A}\otimes{}^{inv}\Gamma$ onto $^{inv}\Gamma\otimes\mathcal{A}$. Their restrictions on this space are given by

$$\sigma^{\mathrm r}_n(\mathrm{id} \otimes \varsigma) = (\varsigma \otimes \mathrm{id})\tau, \tag{A.12}$$

for each $n \in \mathbb{Z}$.

As a right $\mathcal{A}$-module, the space Γ is naturally identificable with $^{inv}\Gamma \otimes \mathcal{A}$. The isomorphism is induced by the multiplication map $m^{\mathrm r}_\Gamma$. Moreover,

$$[m^{\mathrm r}_\Gamma{\restriction}^{inv}\Gamma \otimes \mathcal{A}]^{-1} = (Q \otimes \mathrm{id})\varrho_\Gamma. \tag{A.13}$$

In terms of the structuralization $\Gamma \leftrightarrow {}^{inv}\Gamma \otimes \mathcal{A}$, the following correspondences hold

$$m^{\mathrm r}_\Gamma \leftrightarrow \mathrm{id} \otimes m \qquad \varrho_\Gamma \leftrightarrow \mathrm{id} \otimes \phi \tag{A.14}$$

$$d \leftrightarrow (\varsigma \otimes \mathrm{id})\phi \tag{A.15}$$

$$m^{\mathrm l}_\Gamma \leftrightarrow (\bullet \otimes m)(\mathrm{id} \otimes {}_*\sigma \otimes \mathrm{id})(\phi \otimes \mathrm{id}^2). \tag{A.16}$$

Here, $_*\sigma\colon \mathcal{A} \otimes {}^{inv}\Gamma \to {}^{inv}\Gamma \otimes \mathcal{A}$ is the restriction of the operators $\sigma^{\mathrm r}_n$, and the map $\bullet\colon \mathcal{A} \otimes {}^{inv}\Gamma \to {}^{inv}\Gamma$ is given by

$$a \bullet \vartheta = Q(a\vartheta). \tag{A.17}$$

This map determines a left $\mathcal{A}_0$-module structure on $^{inv}\Gamma$. We have also

$$a \bullet \varsigma(b) = \varsigma m_0(a \otimes b) - \varsigma(a)\epsilon(b). \tag{A.18}$$

The space Γ is also trivial as a left $\mathcal{A}$-module. The corresponding isomorphism $\Gamma \leftrightarrow \mathcal{A} \otimes {}^{inv}\Gamma$ is induced by the product map, and explicitly

$$(m^{\mathrm l}_\Gamma{\restriction}\mathcal{A} \otimes {}^{inv}\Gamma) = (\bullet \otimes \mathrm{id})(\mathrm{id} \otimes {}_*\sigma)(\phi \otimes \mathrm{id})\colon \mathcal{A} \otimes {}^{inv}\Gamma \to {}^{inv}\Gamma \otimes \mathcal{A}$$

$$(m^{\mathrm l}_\Gamma{\restriction}\mathcal{A} \otimes {}^{inv}\Gamma)^{-1} = (\kappa \otimes \bullet)(\phi\kappa^{-1} \otimes \mathrm{id})({}_*\sigma)^{-1}\colon {}^{inv}\Gamma \otimes \mathcal{A} \to \mathcal{A} \otimes {}^{inv}\Gamma.$$

In terms of the structuralization $\Gamma = \mathcal{A} \otimes {}^{inv}\Gamma$ the following correspondences hold:

$$(A.19) \qquad m_\Gamma^{\mathrm{r}} \leftrightarrow [m \otimes \bullet(\kappa^{-1} \otimes \mathrm{id})](\mathrm{id} \otimes \sigma^{-1}\phi \otimes \mathrm{id})(\mathrm{id} \otimes ({}_*\sigma)^{-1})$$

$$(A.20) \qquad m_\Gamma^{\mathrm{l}} \leftrightarrow m \otimes \mathrm{id}$$

$$(A.21) \qquad \varrho_\Gamma \leftrightarrow (\mathrm{id} \otimes {}_*\sigma)(\phi \otimes \mathrm{id})$$

$$(A.22) \qquad -d \leftrightarrow (\kappa \otimes \varsigma)\phi\kappa^{-1} = (\mathrm{id} \otimes \varsigma\kappa^{-1})\sigma^{-1}\phi.$$

The structure of every right-covariant calculus Γ is completely determined by the space $\mathcal{K} = \ker(\varsigma) \cap \ker(\epsilon)$. This space is a left $\mathcal{A}_0$-ideal satisfying

$$(A.23) \qquad \tau(\mathcal{A} \otimes \mathcal{K}) = \mathcal{K} \otimes \mathcal{A}.$$

Conversely, let $\mathcal{K} \subseteq \ker(\epsilon)$ be a left $\mathcal{A}_0$-ideal such that equality (A.23) holds. The space ${}^{inv}\Gamma$ and maps ς and $\bullet$ can be recovered as

$$(A.24) \qquad {}^{inv}\Gamma = \ker(\epsilon)/\mathcal{K} \qquad \varsigma(a) = [a - \epsilon(a)]_\mathcal{K}$$

$$(A.25) \qquad \bullet(a \otimes [b]_\mathcal{K}) = [m\tau^{-1}\sigma(a \otimes b)]_\mathcal{K}.$$

The whole right-covariant calculus Γ is then constructed with the help of the above established correspondences.

Appendix B. Elementary Properties of The Adjoint Action

By definition, the adjoint action of G onto itself is a linear map $\mathrm{ad}\colon \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}$ defined by

$$(B.1) \qquad \mathrm{ad} = (\mathrm{id} \otimes m)(\mathrm{id} \otimes \kappa \otimes \mathrm{id})(\tau \otimes \mathrm{id})(\mathrm{id} \otimes \phi)\phi.$$

Lemma B.1. *The following identities hold*

$$(B.2) \qquad (\mathrm{id} \otimes \epsilon)\mathrm{ad} = \mathrm{id}$$

$$(B.3) \qquad (\mathrm{id} \otimes \phi)\mathrm{ad} = (\mathrm{ad} \otimes \mathrm{id})\mathrm{ad}.$$

In other words, ad *is a counital and coassociative map.*

Proof. We compute

$$(\mathrm{id} \otimes \epsilon)\mathrm{ad} = (\mathrm{id} \otimes \epsilon \otimes \epsilon)(\mathrm{id} \otimes \sigma^{-1}\tau)(\tau \otimes \mathrm{id})(\kappa \otimes \phi)\phi$$

$$= (\mathrm{id} \otimes \epsilon \otimes \epsilon)(\tau \otimes \mathrm{id})(\mathrm{id} \otimes \phi)\sigma^{-1}\tau(\kappa \otimes \mathrm{id})\phi = (\mathrm{id} \otimes \epsilon\kappa)\sigma\phi = \kappa^{-1}(\mathrm{id} \otimes \epsilon)\phi\kappa = \mathrm{id}.$$

Computation of the left-hand side of (B.3) gives

$$(\mathrm{id} \otimes m \otimes m)(\mathrm{id}^2 \otimes \sigma \otimes \mathrm{id})(\mathrm{id} \otimes \phi \otimes \phi)(\tau \otimes \mathrm{id})(\kappa \otimes \phi)\phi =$$
$$= (\mathrm{id} \otimes m \otimes m)(\tau \otimes \sigma \otimes \mathrm{id})(\mathrm{id} \otimes \tau \otimes \mathrm{id}^2)(\phi \otimes \mathrm{id} \otimes \phi)(\kappa \otimes \phi)\phi$$
$$= (\mathrm{id} \otimes m \otimes m)(\tau \otimes \mathrm{id}^3)(\kappa \otimes \mathrm{id} \otimes \sigma \otimes \mathrm{id})(\mathrm{id} \otimes \tau \otimes \mathrm{id}^2)(\mathrm{id} \otimes \kappa \otimes \mathrm{id}^3)(\sigma \otimes \phi \otimes \mathrm{id})(\phi \otimes \phi)\phi$$
$$= (\mathrm{id} \otimes m \otimes m)(\tau \otimes \mathrm{id} \otimes \kappa \otimes \mathrm{id})(\kappa \otimes \mathrm{id} \otimes \tau\sigma^{-1}\tau \otimes \mathrm{id})(\mathrm{id} \otimes \tau \otimes \mathrm{id}^2)(\sigma \otimes \phi \otimes \mathrm{id})(\phi \otimes \phi)\phi$$
$$= (\mathrm{id} \otimes m \otimes m)(\tau \otimes \mathrm{id} \otimes \kappa \otimes \mathrm{id})(\kappa \otimes \mathrm{id}^4)(\phi \otimes \mathrm{id}^3)(\mathrm{id} \otimes \tau\sigma^{-1}\tau \otimes \mathrm{id})(\sigma \otimes \mathrm{id}^2)(\phi \otimes \phi)\phi$$
$$= (\mathrm{id} \otimes m \otimes m)(\tau \otimes \mathrm{id} \otimes \kappa \otimes \mathrm{id})(\kappa \otimes \mathrm{id}^4)(\phi \otimes \mathrm{id}^3)(\phi \otimes \mathrm{id}^2)(\tau \otimes \mathrm{id})(\mathrm{id} \otimes \phi)\phi$$
$$= (\mathrm{ad} \otimes m)(\mathrm{id} \otimes \kappa \otimes \mathrm{id})(\tau \otimes \mathrm{id})(\mathrm{id} \otimes \phi)\phi = (\mathrm{ad} \otimes \mathrm{id})\mathrm{ad},$$

which completes the proof. □

Further useful identities are

Lemma B.2. *We have*

$$(\epsilon \otimes \mathrm{id})\mathrm{ad} = 1\epsilon$$
$$\big[\mathrm{id} \otimes m(\mathrm{id} \otimes \kappa) \otimes \mathrm{id}\big](\mathrm{ad} \otimes \phi)\phi = (\mathrm{id} \otimes \kappa \otimes \mathrm{id})(\tau \otimes \mathrm{id})(\mathrm{id} \otimes \phi)\phi.$$

Proof. Let us check the second identity. A direct computation gives

$$\big[\mathrm{id} \otimes m(\mathrm{id} \otimes \kappa) \otimes \mathrm{id}\big](\mathrm{ad} \otimes \phi)\phi(a) =$$
$$= \big(\mathrm{id} \otimes m(\mathrm{id} \otimes \kappa) \otimes \mathrm{id}\big)(\mathrm{id} \otimes m \otimes \mathrm{id}^2)(\kappa \otimes \mathrm{id}^4)(\tau \otimes \mathrm{id}^3)(a^{(1)} \otimes a^{(2)} \otimes a^{(3)} \otimes a^{(4)} \otimes a^{(5)})$$
$$= (\mathrm{id} \otimes m \otimes \mathrm{id})\big(\tau \otimes m(\mathrm{id} \otimes \kappa) \otimes \mathrm{id}\big)(\kappa(a^{(1)}) \otimes a^{(2)} \otimes a^{(3)} \otimes a^{(4)} \otimes a^{(5)})$$
$$= (\mathrm{id} \otimes m \otimes \mathrm{id})(\tau \otimes \mathrm{id}^2)(\kappa(a^{(1)}) \otimes a^{(2)} \otimes 1\epsilon(a^{(3)}) \otimes a^{(4)})$$
$$= (\mathrm{id} \otimes m \otimes \mathrm{id})(\mathrm{id} \otimes \kappa \otimes \mathrm{id})(\tau \otimes \mathrm{id})(\mathrm{id} \otimes \phi)\phi(a). \quad □$$

Finally, let us study the twisting properties of the adjoint action.

Lemma B.3. *The following identities hold*

(B.4) $$\qquad (\mathrm{id} \otimes \mathrm{ad})\sigma_m = (\sigma_m \otimes \mathrm{id})(\mathrm{id} \otimes \sigma_n)(\mathrm{ad} \otimes \mathrm{id})$$
(B.5) $$\qquad (\mathrm{ad} \otimes \mathrm{id})\sigma_n = (\mathrm{id} \otimes \sigma_m)(\sigma_n \otimes \mathrm{id})(\mathrm{id} \otimes \mathrm{ad}).$$

Proof. We compute

$$(\sigma_m \otimes \mathrm{id})(\mathrm{id} \otimes \sigma_n)(\mathrm{ad} \otimes \mathrm{id}) =$$
$$= (\sigma_m \otimes \mathrm{id})(\mathrm{id} \otimes \sigma_n)(\mathrm{id} \otimes m \otimes \mathrm{id})(\tau \otimes \mathrm{id}^2)(\kappa \otimes \mathrm{id}^3)\big((\phi \otimes \mathrm{id})\phi \otimes \mathrm{id}\big)$$
$$= (\sigma_m \otimes m)(\mathrm{id} \otimes \sigma_n \otimes \mathrm{id})(\tau \otimes \sigma_n)(\kappa \otimes \mathrm{id}^3)\big((\phi \otimes \mathrm{id})\phi \otimes \mathrm{id}\big)$$
$$= (\mathrm{id}^2 \otimes m)(\mathrm{id} \otimes \tau \otimes \mathrm{id})(\mathrm{id} \otimes \kappa \otimes \mathrm{id}^2)(\sigma_{-n} \otimes \mathrm{id}^2)(\mathrm{id} \otimes \sigma_m \otimes \mathrm{id})(\mathrm{id}^2 \otimes \sigma_n)\big((\phi \otimes \mathrm{id})\phi \otimes \mathrm{id}\big)$$
$$= (\mathrm{id}^2 \otimes m)(\mathrm{id} \otimes \tau \otimes \mathrm{id})(\mathrm{id} \otimes \kappa \otimes \mathrm{id}^2)\big(\mathrm{id} \otimes (\phi \otimes \mathrm{id})\phi\big)\sigma_m = (\mathrm{id} \otimes \mathrm{ad})\sigma_m.$$

Furthermore, we have

$$(\mathrm{id}\otimes\sigma_m)(\sigma_n\otimes\mathrm{id})(\mathrm{id}\otimes\mathrm{ad}) = (\mathrm{id}\otimes\sigma_m)(\sigma_n\otimes m)(\mathrm{id}\otimes\tau\otimes\mathrm{id})(\mathrm{id}\otimes\kappa\otimes\mathrm{id}^2)\big(\mathrm{id}\otimes(\phi\otimes\mathrm{id})\phi\big)$$

$$= (\mathrm{id}\otimes m\otimes\mathrm{id})(\mathrm{id}^2\otimes\sigma_m)(\mathrm{id}\otimes\sigma_m\otimes\mathrm{id})(\sigma_n\otimes\mathrm{id}^2)(\mathrm{id}\otimes\tau\otimes\mathrm{id})(\mathrm{id}\otimes\kappa\otimes\mathrm{id}^2)\big(\mathrm{id}\otimes(\phi\otimes\mathrm{id})\phi\big)$$

$$= (\mathrm{id}\otimes m\otimes\mathrm{id})(\tau\otimes\sigma_m)(\mathrm{id}\otimes\sigma_n\otimes\mathrm{id})(\sigma_m\otimes\mathrm{id}^2)(\mathrm{id}\otimes\kappa\otimes\mathrm{id}^2)\big(\mathrm{id}\otimes(\phi\otimes\mathrm{id})\phi\big)$$

$$= (\mathrm{id}\otimes m\otimes\mathrm{id})(\tau\otimes\mathrm{id}^2)(\kappa\otimes\mathrm{id}^3)(\mathrm{id}^2\otimes\sigma_m)(\mathrm{id}\otimes\sigma_n\otimes\mathrm{id})(\sigma_{-m}\otimes\mathrm{id}^2)\big(\mathrm{id}\otimes(\phi\otimes\mathrm{id})\phi\big)$$

$$= (\mathrm{id}\otimes m\otimes\mathrm{id})(\tau\otimes\mathrm{id}^2)(\kappa\otimes\mathrm{id}^3)\big((\phi\otimes\mathrm{id})\phi\otimes\mathrm{id}\big)\sigma_n = (\mathrm{ad}\otimes\mathrm{id})\sigma_n. \quad \square$$

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